

# Networking of technologies: a dialog between CAS and DGS

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## Abstract

*Different technologies such as Computer Algebra Systems (CAS) and Dynamical Geometry Systems (DGS) have been developed along the last decades. CAS have high abilities for symbolic computations and also graphical features, and DGS are characterized by their interactivity and exploratory abilities. Due to the implementation of CAS features, DGS are now equipped with algorithms and commands which enable automated exploration and proof. For numerous situations, the symbolic strength of a CAS is needed besides the DGS, and a dialog must be installed between them. In this paper, we show the efficiency of this dialog, which we call networking of technologies: it leads to conjectures and enables to verify them. It may lead to new understanding, beyond the first conjecture. The starting point (the artifact) of a new instrumental genesis is now a pair DGS-CAS.*

## 1 Introduction

The development of mathematics has always been influenced by the development of new techniques and technologies. Trouche [39] describes different steps of the evolution. In particular, he quotes Lavoie's claim [30] that the revolution made by the introduction of the electronic calculator is equivalent to the introduction of the iron quill instead of the feather to write. According to Lavoie, the new technology changes the man-machine interaction and task sharing; it frees the user from pure technicalities and incites the him/her to develop more profound understanding. Other steps were the introduction of slider rulers, and at the beginning of the 1970's the appearance of electronic calculators. At first only numerical algorithms were available. Along the last decades, algorithms for symbolic computations have been developed. We emphasize Bruno Buchberger's Ph.D. thesis (1965), where he defined the so-called Gröbner bases. He chose this name to honor his advisor. Gröbner bases and the so-called Buchberger's algorithm are the basis for automatic solution of nonlinear polynomial systems of equations [12, 13]. Buchberger's algorithm—which is based on a kind of rediscovery of Günther's

work from 1913 and Hironaka's concept from 1964—became very popular and crucial for Computer Algebra Systems (CAS). Several contributions and improvements were published. For example, an essential advance has been made with the Gröbner Cover, enabling the solution of parametric systems of nonlinear polynomial equations [33].

These algorithms have been implemented in CAS, either free or commercial, either general (Maple, Mathematica, etc.) or specific (CoCoA, Singular, etc.). Other algorithms have been implemented to solve non-polynomial system of equations.

Another evolution has been made with developments of Dynamic Geometry Systems (DGS), either free or commercial (Geometer Sketchpad, Cabri, GeoGebra, etc.). A DGS is a kind of software enabling construction of geometric objects and interactive manipulations. In what they call spatiovisual reasoning, three important developments are highlighted in [37] in mathematics education, one of them is that “reasoning involves thinking about, and making decisions based on, visuospatial perception and understanding, both of which are influenced by prior knowledge and context of learning.” The next development is that “visuospatial reasoning in geometry can be improved through experience from perception to higher levels of reasoning.” In the present paper, we relate to 2D situations, but these remarks are valid for 3D also.

New technological developments are always on their way [36]: CAS engines have been implemented into DGS, for example Giac in GeoGebra (see [28]).

As for any tool, the user has to discover the abilities of his/her CAS and DGS. During this process, called *instrumental genesis*, the user develops his/her own schemes. The abilities of the software influence the user's thinking. Artigue says in [4]: “Professionals accept that there is a cost to learning how to use such software effectively”. The process is called instrumentation: the tool (called artifact) is transformed into an instrument by a process, which may be different from one user to the other. Each user develops personal understanding while acquiring the abilities of the software. Actually, the user is also influenced by his/her discovery of new features of the software. Roughly speaking, by this instrumental genesis, the artifact together with the user's mental schemes and usage skills becomes an instrument. The new knowledge, including the CAS/DGS literacy is now integral part of the mathematical knowledge of the user. See also [24, 4, 34].

In mathematics education, pure technical skills are not the goal. Artigue says [4]: “What is aimed at by mathematics education, and especially by general mathematics education in school and university, is not an efficient mathematical practice assisted by the currently available computational tools; rather, it is concerned with the transmission of the bases of ‘mathematical culture’”. The mathematical culture changes with the new developments and the new available environments. Not only some skills are transferred from the brain to the hand, as mentioned by Lavoie in [30], but new skills are developed and new understandings appear. The always increasing abilities of DGS, in particular when the DGS has a CAS component, lead to the development of a new domain of mathematical developments: automated exploration, automated discovery, automated proofs. This is spectacular for the computation of geometric loci and envelopes (see [1, 2, 16, 17, 8, 10, 29]). The study of isoptic curves had also benefit from such developments of the DGS, but it required also symbolic computations with a separate CAS (see [15, 18]).

In this paper, we show examples and illustrate a new step in the instrumental genesis of the combined use of CAS and DGS. It happens that the user performs two separate processes of instrumental genesis, one for a CAS and one for a DGS, a combined process takes place where the initial tool is the pair DGS-CAS. Using a fruitful dialog between the two ICTs, he/she can transform the pair into a

more complex instrument. We call this process networking of technologies, as a similar process can be built with other kinds of technology.

## 2 Envelopes of families of plane curves

### 2.1 Developments of automated proofs

Consider a parameterized family  $\mathcal{F}$  of plane curves  $\mathcal{C}_k$ , dependent on a real parameter  $k$ . A plane curve  $\mathcal{E}$  is called an envelope of the family  $\mathcal{F}$  if the following properties hold: (i) every curve  $\mathcal{C}_k$  in  $\mathcal{F}$  is tangent to  $\mathcal{E}$ ; (ii) to every point  $M$  on  $\mathcal{E}$  is associated a value  $k(M)$  of the parameter  $k$ , such that the curve  $\mathcal{C}_k$  is tangent to  $\mathcal{E}$  at the point  $M$ ; (iii) The function  $k(M)$  is non-constant on every arc of  $\mathcal{E}$ .

In [26], Kock calls this definition an *impredicative definition*. He gives two other (equivalent) definitions.

A more applicable definition is to consider “infinitely close” curves in the family and then computing a limit. The drawback of this method is the imprecision of the notion of limit, as it requires the definition of a topological space of curves. A computational definition is given by the following theorem (see [16] for a short proof).

**Theorem 1** *Suppose that the equation of the curve  $\mathcal{C}_k$  is  $F(x, y, k) = 0$ . If it exists, the envelope  $\mathcal{E}$  is defined by the solution of the system of equations:*

$$\begin{cases} F(x, y, k) = 0, \\ \frac{\partial F}{\partial k}(x, y, k) = 0. \end{cases} \quad (1)$$

Note that is the definition of an envelope of a family of plane curves in Berger’s book [6]. Kock [26] calls it an *analytic definition*.

The immediate consequence is the necessity to have computational methods in order to solve a system of equations. In a general CAS, there exists a solve command for this purpose. Actually, this command works in different ways for different kinds of equations. For algebraic curves, i.e. when  $F$  is a polynomial, methods based on the theory of Gröbner bases are used [33]. Pattern recognition is needed for the command to decide how to proceed. We will see in Section 3 that even for polynomial data, the determination of the envelope may not be polynomial, even not rational—simply because non-rational methods can be helpful in several cases. (On the other hand, polynomial methods can always compute the output that is also polynomial, at least theoretically.) Automated methods for the determination of envelopes have been implemented in DGS: GeoGebra, Cabri, etc. The first steps in this implementation are described in [1, 8], including explanations on false components in the output. Further developments took place and the problem of false components solved. This, together with a complete exposition of the mathematics involved, is given by [10], introducing the notion of acceptable special component. They work in the setting of Polynomial Ideals, and the main part is elimination of some variables. This has been implemented as the command **Envelope** in GeoGebra version 5. Their developments include more complicated cases.

Numerous examples of envelopes using together the DGS GeoGebra and a CAS (at first Derive, then mostly Maple) are studied in [16, 17]. Sometimes, the examples under study involved trigonometric functions; using well-known formulas, sines and cosines have been replaced by rational expressions. In a further step, these equations have been transformed into polynomial equations. In

the next section, we study the dialog between the two kinds of software needed for dealing with the determination of envelopes. We describe two examples: in one of them the DGS driven exploration may lead to an intuition that will reveal incorrect after the analytic work with the CAS.

## 2.2 Concrete meanings of envelopes

Envelopes of parametric families of plane curves (and even more than that) of parametric families of space curves and parametric families of surfaces in 3-dimensional space have numerous applications. For example, Figure 1 shows a sprinkler: the envelope of the water drops trajectories determines the zone where the soil will remain dry. Another important example is met in industrial plants: robots at



Figure 1: A sprinkler

work may be dangerous for human beings in their vicinity. Therefore, the allowed paths are marked on the floor by either yellow or red lines. They are generally determined by the envelopes of the trajectories of the robotic parts. For the sake of simplicity, these marking lines are approximated by straight lines; see Figure 2.

In Section 3, we show examples where this interpretation, despite being intuitive, is not always correct. In these cases, Sendra et al. make a distinction between envelopes and what they call *offset* (see [35]).



Figure 2: Floor marking in an industrial plant

Already in 1962, Thom laments about the disappearance of envelopes from the standard syllabus; see [38]. He notes that maybe the problem is that the theory of envelopes has not enough theorems, and the study of envelopes requires the study of too many special cases. On the other hand, as we saw above, envelopes describe familiar situations, which could have no explanation without it. “It is

enough to look at a hemispheric cup of coffee with suitable lighting to see the cuspidal structure of a caustic and its variations when lighting changes”. Two examples are shown in next Section.

A recent GeoGebra applet <https://www.geogebra.org/m/drnyhBFV#material/q4xHZwnX> shows an example of trajectories of the working head of a mechanical excavator. The trajectories are built following the action of several sliders, which command the three rotation axes of the machine. The safety zone can be understood. Note that in this example, the family of trajectories depends on 3 parameters, a more complicated situation than what we show in next section, where we work with 1-parameter families of curves.

### 3 A dialog between DGS and CAS: two examples

#### 3.1 First example: unit circles centered on an ellipse

We consider the family of unit circles centered on the ellipse whose canonical equation is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1. \quad (2)$$

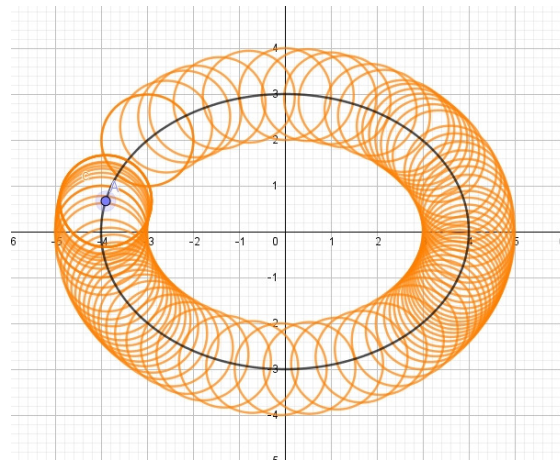


Figure 3: First DGS exploration of a family of circles centered on an ellipse

Figure 3 shows the ellipse and circles in the family. The figure has been obtained with the **Move** command of GeoGebra, with the **TraceOn** option, i.e. the point  $A$  (the center of the circle) is moved along the ellipse using the mouse, and the circles remain on the plot.

As a first intuition, the user guesses that there are two envelopes, shaped as ellipses. An intuition, not a proof. The proof has to be performed using analytic methods.

The ellipse has a parametric presentation, namely

$$\begin{cases} x(t) = 4 \cos t, \\ y(t) = 3 \sin t, \end{cases} \quad t \in [0, 2\pi].$$

Whence the following general equation for the circles of the family is under study:

$$(x - 4 \cos t)^2 + (y - 3 \sin t)^2 - 1 = 0.$$

We denote by  $F(x, y, t)$  the left-hand side of this equation. According to Theorem 1, an envelope of the family of circles is determined by the following system of equations:

$$\begin{cases} (x - 4 \cos t)^2 + (y - 3 \sin t)^2 - 1 = 0, \\ 8(x - 4 \cos t) \sin t - 6(y - 3 \sin t) \cos t = 0. \end{cases} \quad (3)$$

The analytic work can be performed by a CAS (as said previously, we use Maple [32]—but for this task GeoGebra’s built-in CAS can also be used, see <https://www.geogebra.org/m/ubbsnbu7> for more details).

The solutions of Equations (3) are as follows:

$$\begin{cases} x(t) = \frac{\cos t (4\sqrt{16-7 \cos^2 t}+3)}{\sqrt{16-7 \cos^2 t}}, \\ y(t) = \frac{\sin t (3\sqrt{16-7 \cos^2 t}+4)}{\sqrt{16-7 \cos^2 t}} \end{cases} \quad (4)$$

and

$$\begin{cases} x(t) = \frac{\cos t (4\sqrt{16-7 \cos^2 t}-3)}{\sqrt{16-7 \cos^2 t}}, \\ y(t) = \frac{\sin t (3\sqrt{16-7 \cos^2 t}-4)}{\sqrt{16-7 \cos^2 t}} \end{cases} \quad (5)$$

**Remark 2** *Using two different CAS, we obtained expressions looking really different. Algebraic manipulations enabled to show that the expressions define actually the same functions.*

The output is non-polynomial; it has been copied into the algebraic window of GeoGebra, as shown in Figure 4. Equations (a) in the algebraic window (on the left side), which are actually Equations (4) above, describe the external component. Equations (b) in the algebraic window, which are a copy of Equations (5), describe the internal component. Follow the colors in Figure 4.

The two components are closed loops. They look smooth, actually they look like ellipses, but it is impossible to prove this by looking at the parametric equations only. An experimental approach is to use the GeoGebra command to plot a conic through 5 given points. With suitable zooming, the difference between each loop and ellipses appears. Of course, the accuracy of such an experimental proof is conditioned by a rigorous placement of the points (using the **Point on Object** command). Here a strong **Zoom In** is necessary to identify the slight difference; see Figure 5, where the difference appears on the left side.

A more rigorous way is as follows: it can be easily shown that the points  $(-3, 0)$ ,  $(3, 0)$ ,  $(0, 2)$  and  $(0, -2)$  belong to the inner loop. If this inner loop is an ellipse, then it has equation  $\frac{x^2}{9} + \frac{y^2}{4} = 1$ . By plotting this ellipse and comparing with the inner loop obtained previously, the graphical difference is clearer. See Figure 5. An analytic proof can also be written: substitute the parametric expressions of  $x(t)$  and  $y(t)$  into the implicit equation of the ellipse and check that they do not verify the equation. A similar consideration can be made for the outer loop—we obtain  $\frac{x^2}{25} + \frac{y^2}{16} = 1$  for the implicit equation.

For another rigorous proof of falsity, sines and cosines should be replaced by rational expressions. By successive squaring, the square roots can be removed. We give some details of this method for the example in next subsection.

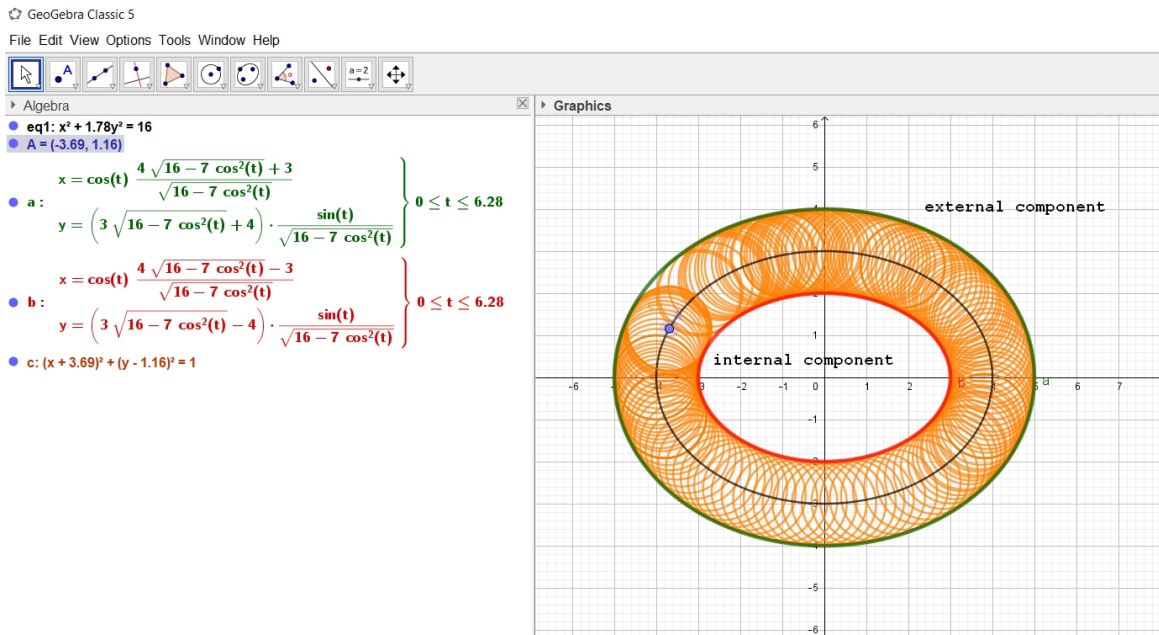


Figure 4: Envelope of a family of circles centered on an ellipse

However, we provide here a different way to obtain the correct implicit equation for the union of the two loops. Surprisingly, there will be no two implicit algebraic equations, just one—it characterizes both loops at the same time.

To achieve this, we consider again the implicit equation (2) and add the equation of the circle

$$(u - x)^2 + (v - y)^2 = 1 \tag{6}$$

where  $u$  and  $v$  are the coordinates of an arbitrary point on the circle. By following the technique described in [9] we obtain that

$$\det \begin{pmatrix} \frac{2x}{16} & \frac{2y}{9} \\ -2u + 2x & -2v + 2y \end{pmatrix} = 0 \tag{7}$$

must hold—this describes the differential condition of (1) for implicit systems. By eliminating  $x$  and  $y$  from the equation system (2), (6), (7) we obtain the equation

$$81u^8 + 450u^6v^2 + 913u^4v^4 + 800u^2v^6 + 256v^8 - 3762u^6 - 13400u^4v^2 - 12770u^2v^4 - 1760v^6 + 55633u^4 + 82090u^2v^2 - 39791v^4 - 333424u^2 - 5460v^2 + 705600 = 0 \tag{8}$$

whose left hand side is an octic polynomial and it can be checked—again, by using a CAS—that it is not a multiple of any quadratic polynomials whose plots could be similar to the loops. We remark that by using Maple’s `evala(AFactor(...))` command we can verify that (8) is irreducible over  $\mathbb{C}$ , so the loops cannot be expressed separately as implicit algebraic equations. (The same result can also be achieved by using Singular’s [21] absolute factorization library (`absfact_lib`) [11, 7] that is freely available for schools.)

A DGS version of the problem, that uses a mixture of symbolic and numerical techniques, can be found at <https://www.geogebra.org/m/xvddq4xq>. The applet, as of June 2020, can find that sought implicit equation is an octic, but it cannot print its simplest form as given in (8).

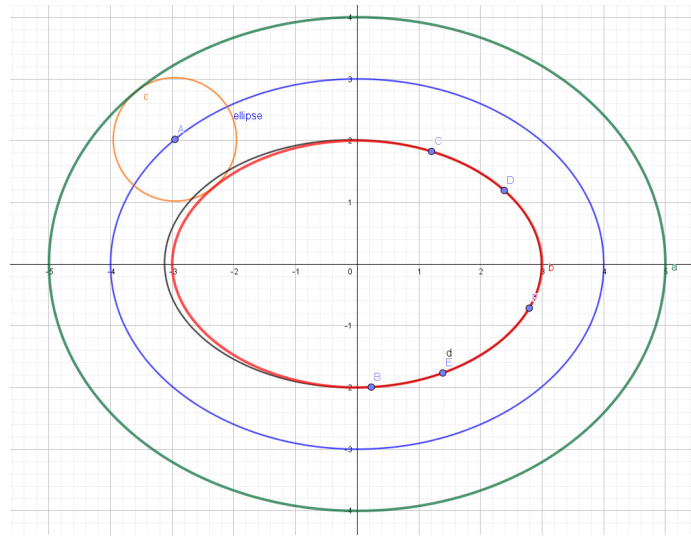


Figure 5: The internal loop is not an ellipse—experimental proof

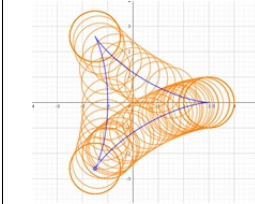
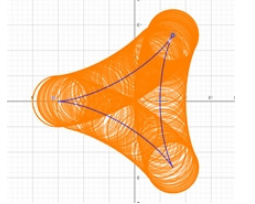
### 3.2 Second example: circles centered on a deltoid

We consider the deltoid whose parametric presentation is

$$\begin{cases} x(t) = 2 \cos t + \cos 2t, \\ y(t) = 2 \sin t - \sin 2t, \end{cases} \quad t \in [0, 2\pi].$$

and the family of unit circles centered on this deltoid. Table 1, with its Figures, shows a DGS assisted exploration.

Table 1: Circles centered on a deltoid—DGS exploration

		
<b>Figures</b>		
<b>Moves of point A</b>	Quick moves	Slow moves
<b>Intuition for an envelope</b>	One component “around” the circles	Something appears in the interior
<b>Open question</b>		What really happens inside?

A general equation for unit circles centered on the deltoid is

$$(x - (2 \cos t + c \cos 2t))^2 + (y - (2 \sin t - \sin 2t))^2 = 1.$$



Therefore, according to Theorem 1, an envelope is determined by the solutions of the system of equations:

$$\begin{cases} (x - (2 \cos t + c \cos 2t))^2 + (y - (2 \sin t - \sin 2t))^2 = 1, \\ 2(x - 2 \cos t - \cos 2t)(2 \sin t + 2 \sin 2t) + 2(y - 2 \sin t + \sin 2t)(-2 \cos t + 2 \cos 2t) = 0. \end{cases}$$

The solution is given by the two following parametric presentations:

$$\begin{cases} x(t) = \frac{1-\cos t}{2 \sin t} \sqrt{2+2 \cos t} + 4 \sin t \left( \cos^2 t + \cos t - \frac{1}{2} \right), \\ y(t) = -\sin 2t + 2 \sin t + \frac{1}{2} \sqrt{2+2 \cos t}, \end{cases} \quad (9)$$

and

$$\begin{cases} x(t) = \frac{\cos t-1}{2 \sin t} \sqrt{2+2 \cos t} + 4 \sin t \left( \cos^2 t + \cos t - \frac{1}{2} \right), \\ y(t) = -\sin 2t + 2 \sin t - \frac{1}{2} \sqrt{2+2 \cos t}. \end{cases} \quad (10)$$

The first output, obtained with the **solve** command, was more complicated and, of course, we used the **simplify** command. This is the output that we copy-pasted into GeoGebra. Note that with respect to the variables  $x$  and  $y$ , the equations are polynomial, but not with respect to the variable  $t$ . A snapshot of a GeoGebra session is displayed in Figure 6.

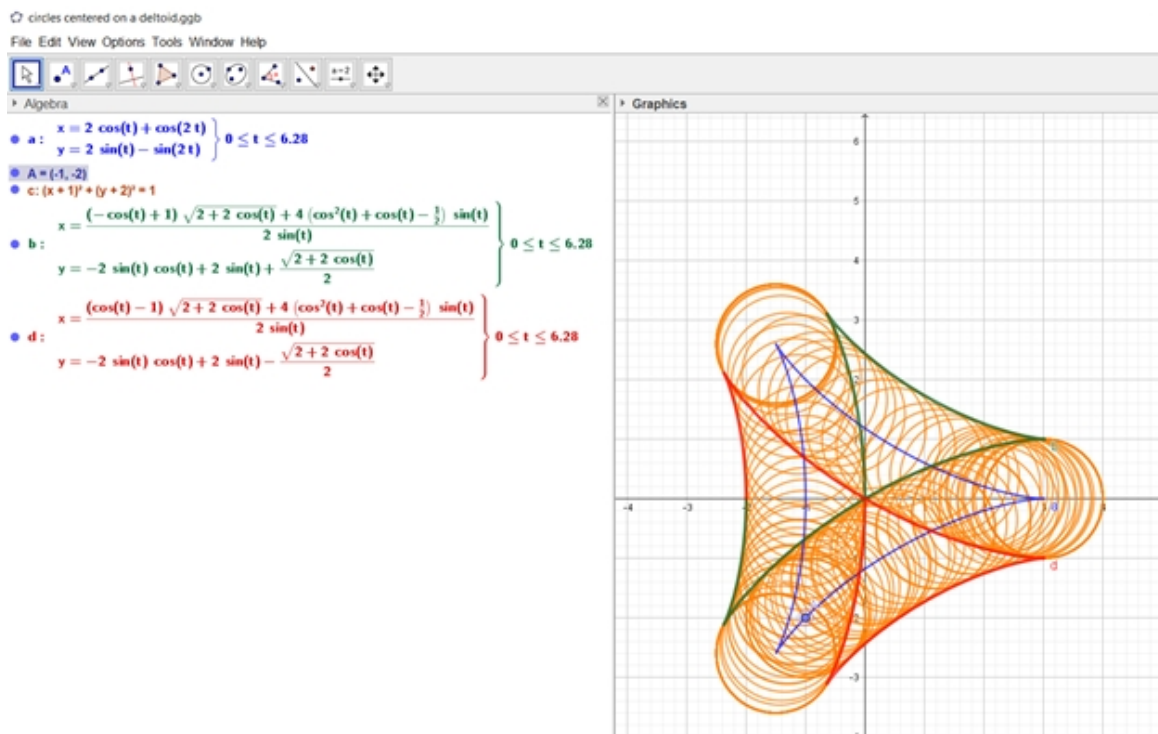


Figure 6: Interactive exploration of the envelope and its components

Two components appear. Both are open curves, mostly inside the area covered by the circles. According to the definition, both components and their union, can be considered as envelopes of the family of unit circles. Let us analyze Figure 7. The Maple code is given in Section 5. When the point

$A$  moves along the deltoid, the unit circle centered at  $A$  is tangent at one point in each component determined by the analytic computations. Actually, this can be viewed in a better way using an animation provided by the CAS. Figure 7 shows two snapshots of such an animation. The value of the parameter  $t$  defining the point  $A$  appears above the plot, and in each case, the two tangency points can be clearly identified. Note that when  $A$  is at a cusp of the deltoid, then the points of contacts with the envelope are at cusps also, as shown in Figure 7 (a). A proof uses derived vectors and checks that the same values of the parameter correspond to both cusps of the deltoid and cusps of the envelope.

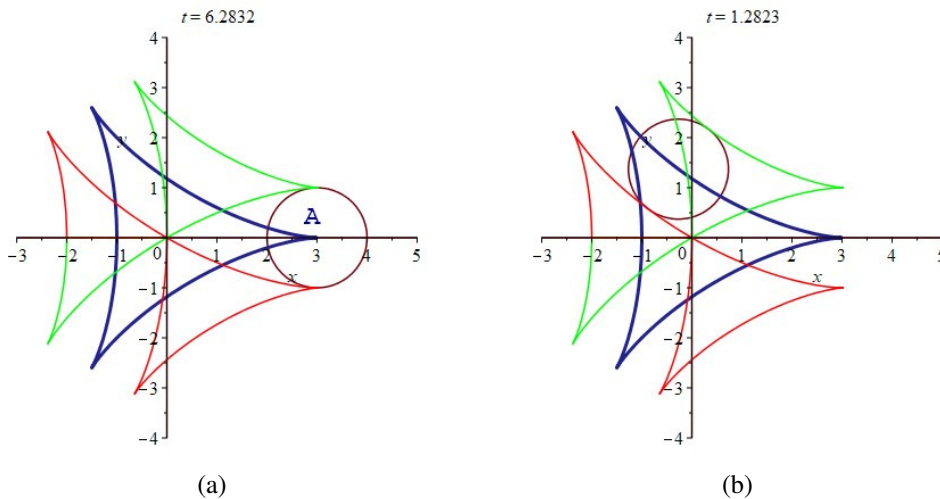


Figure 7: Snapshots of a Maple animation

After having obtained parametric presentations for the envelope, it is important to find an implicit equation, if possible. A possible method is to substitute rational expressions for sine and cosine, based on the following remark. The unit circle centred at the origin has the following parametrizations:

- a trigonometric one:

$$\begin{cases} x = \cos t, \\ y = \sin t, \end{cases}$$

- a rational one:

$$\begin{cases} x = \frac{1-u^2}{1+u^2}, \\ y = \frac{2u}{1+u^2}. \end{cases}$$

It follows that, for almost every  $t$ , there exists a real  $u$  such that

$$\begin{cases} \cos t = \frac{1-u^2}{1+u^2}, \\ \sin t = \frac{2u}{1+u^2}. \end{cases} \quad (11)$$

For more details, we refer to [35], Chapter 4.

We substitute the rational expressions of Equations (11) into Equations 9, using back the variable  $t$  as the parameter (only to fit the tradition). We obtain:

$$\begin{cases} x(t) = \frac{1}{(t^2+1)^2} \left( (t^2+1)^{3/2} - t^4 - 6t^2 + 3 \right), \\ y(t) = (t^2+1)^{3/2} + \frac{8t^3}{(t^2+1)^2}. \end{cases}$$

In both equations, we multiply by a least common denominator, square both sides, then transfer from side to side. Thus, we obtain two polynomials:

$$\begin{cases} P_1(x, y, t) = (x(t^2 + 1)^2 + t^4 + 6t^2 - 3)^2 - t^2(t^2 + 1)^3, \\ P_2(x, y, t) = (y(t^2 + 1)^2 - 8t^3)^2 - (t^2 + 1)^3. \end{cases} \quad (12)$$

These polynomials generate an ideal in  $\mathbb{R}[x, y, t]$ . We eliminate the parameter  $t$  using Maple's command **EliminationIdeal** and obtain an ideal generated by a polynomial  $F(x, y)$  of degree 16 in two variables  $x$  and  $y$ . In factorized form we have  $F(x, y) = F_1(x, y) \cdot F_2(x, y)$ , where:

$$\begin{aligned} F_1(x, y) = & x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8 - 16x^7 + 16x^5y^2 + 80x^3y^4 \\ & + 48xy^6 + 99x^6 - 279x^4y^2 + 681x^2y^4 + 35y^6 - 216x^5 + 432x^3y^2 \\ & + 648xy^4 - 243x^4 - 486x^2y^2 - 243y^4 + 1458x^3 - 4374xy^2 \end{aligned}$$

and

$$\begin{aligned} F_2(x, y) = & x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8 - 16x^7 + 16x^5y^2 + 80x^3y^4 \\ & + 48xy^6 + 117x^6 - 353x^4y^2 + 607x^2y^4 + 53y^6 - 398x^5 + 284x^3y^2 \\ & + 650xy^4 + 392x^4 + 208x^2y^2 - 216y^4 + 1584x^3 - 2704xy^2 - 3248x^2 + 144y^2 + 1568x. \end{aligned}$$

An implicit plot of  $F_1(x, y) = 0$  coincides with the parametric plot of the desired envelope, obtained previously either with the CAS or with the DGS (Figures 7 and 6). An implicit plot of  $F_2(x, y) = 0$  is irrelevant to the question under study. This can be checked graphically. Such an irrelevant plot appears because of the two squarings which have been performed in order to have a polynomial setting. More details on irrelevant components can be found in [8, 10].

The same process beginning with Formulas (10) provides exactly the same result.

Here we remark that the elimination process described above cannot be easily performed with some freely available CAS including Singular, CoCoA [3] and Giac, so it is difficult to change Maple to a different system here.

## 4 The necessity of networking between technologies

### 4.1 Animation vs dynamic exploration

We wish to compare the effect of the two ways to study the related movements of the point  $A$  and of the points of contact of the corresponding unit circles with the envelope. Table 1 showed snapshots of an interactive exploration using GeoGebra. This was the first part of the work. The user plotted only the deltoid, the point  $A$  and a circle. Then using the mouse, he/she moved the point  $A$  around. No explicit computation, but observation of what appears on the screen. Figure 7 shows snapshots of an animation prepared with Maple. Most of the moves are pre-defined by the programming. After the plot appears on the screen, the user can master the moves only at a certain extent: he/she can change the pace (defining the number of frames per second), even go forth and back using specific arrows. In such an animation, the user sees one moving circle, without tracing it. Note that such an animation can be programmed only after the analytic computations have been performed, as the

Table 2: Circles centered on a deltoid—DGS exploration

	<b>DGS exploration</b>	<b>CAS animation</b>
<b>Plot</b>	No computation needed	Computations are a pre-requisite
<b>User action</b>	Interactive using the mouse	The user is more passive
<b>Trace</b>	Yes	No
<b>Automated determination (Envelope command)</b>	Yes (under limitations of the software)	No
<b>Enables to express conjectures</b>	Yes	No

parametrizations of the envelope components are to be used. Each way of working has its pros and cons. They are briefly described in Table 2.

The second row (Plot) in Table 2 helps to understand the need for networking between the DGS and the CAS: the graphical exploration with the DGS does not give a final answer. Symbolic computations have to be performed, and for this the strength of the algorithms implemented in a CAS is necessary. The symbolic abilities of GeoGebra have not been designed for complicated equations as we have in our examples. We discuss that more in the next subsection. Note that the experience of double visualization, each one with its own features, yielded access to higher level of reasoning, as described by [37]. This is especially true, when switching from the mathematical meaning to physical meaning of the situation: with this double dynamical visualization, it is possible to understand that, if we imagine the situation as a physical device made of a circular object whose center travels along a deltoid, the envelope of the given circles does not describe a frontier for a safety zone. The frontier should be rather described as an offset, in the sense of [35].

## 4.2 Networking

Computation abilities are generally higher with a CAS than with a DGS, and interactive plotting is a characteristic of a DGS. Using Duval's terminology [22] (for presentation of the foundations of his theory) and [23], we say that a CAS enables switching from the algebraic register of representation to the graphical register and a DGS enables switching in both directions.

Duval emphasizes that mathematical objects are not graspable as are the objects of study in Physics, Chemistry, etc. They are graspable only through various representations. The possibility of a study is conditioned by the kind of representation. Duval's first example [22] is the variable cost of numerical computations, depending on the numeration system which has been chosen.

In our study here, working on a special problem related to plane (either algebraic or not) curves, the constant evolution during the last decades of the registers of representation offered by various technologies has made possible a revival of the study of envelopes, a long time after Thom lamented about their disappearance from the syllabus [38]. The same phenomenon is true for the study of isoptic curves, which began in the second half of the 19th century, and see a strong interest nowadays (e.g., see [19, 20]). This process has been made possible as computers became more and more powerful. It is well known that graphical representations, plots of functions and other objects, request strong computational abilities (after all, the CPU has to compute in the background a great number of points

in a numerical representation, then it can plot). The situation presented in this paper requires the usage of two different kinds of software. The first one (the DGS) is appropriate for automated exploration, using dynamical graphics relying on numerical computations in the background, and may lead to a conjecture. We saw in Table 1 that several trials may be necessary to obtain a “reasonable conjecture”. After all it is well known that what you see on the screen and what actually “exists” are not always identical [25, 14]. In order to check the conjecture, the data had to be transferred to a CAS, and there to perform the algebraic computations. In a reversed direction, the analytic results provided by the CAS had to be transferred back to the DGS. Finally, the plot obtained by the DGS using the output of the CAS has been checked and fitted the conjecture. Work may have been performed in reversed direction, first symbolic work with the CAS and the animation it allows. We mentioned previously that the determination of envelopes is often needed to determine safety zones. Our examples show that here the borderline of the safety zone is not the envelope which has been determined. The DGS exploration shows this fact more clearly than the CAS animation (Table 1). The transfer between the two kinds of technology has been actually performed using a low profile method, namely Copy-Paste. This is a minimal level of networking between technologies. We hope that, in the near future, easier ways of dialog between the two environments will be developed.

In the past, students and educators used to discover one technology at a time, hand-held calculator or PC based software. As in every cognitive spiral, the end of a loop is the starting point of a new loop. In our study, the pair DGS-CAS is the tool, with which a new instrumental genesis will be performed to transform the pair into one more powerful instrument. The new instrumental skills are then a new part of the mathematical culture of the user. A good example of the importance of this new instrument is the revival of the classical theory of envelopes, and its modern appearance with automated methods. We foresee other important developments in the next future. Finally, we note that working with this double-instrument requires a new technological discourse [4]. Because of the difference between mathematical meaning of the envelope under study and its uselessness for a concrete implementation and determination of a safety zone, this discourse must be supplemented by teacher’s common sense [5].

At last we highlight that some software systems are making improvements towards supporting a kind of a mixture of CAS/DGS in their user interface. For example, Mathematica’s [40] **Manipulate** command is clearly a step forward—it allows arbitrary Mathematica commands to be embedded in a graphics window by allowing a direct connection between the sliders and the plot (see [31] for a detailed example). In fact, GeoGebra’s **Envelope** command is also a careful combination of using various numerical and symbolic techniques [27], by optimizing the result to get a good balance between exact solution (that may be difficult to interpret) and visual output (that should be easier to overview). These examples support that networking of technologies, even internally, are not avoidable in some advanced research.

## 5 Supplementary Electronic Materials

- The hyperlinks to the GeoGebra applets that we used in this work are inserted in the text.
- Figure 7 is an [animated GIF](#) of a Maple animation. For the sake of the reader, we include here the source code of the animation.

```

restart: with(plots): setoptions(scaling = constrained):
axes := implicitplot({x = 0, y = 0}, x = -3 .. 5, y = -4 .. 4):
F := (x + (-2*cos(t) - cos(2*t)))^2 + (y + (-2*sin(t) + sin(2*t)))^2 - 1:
derF := diff(F, t):
solve({F = 0, derF = 0}, {x, y}):
env := allvalues(%):
# parametrization of 1st component - 1st envelope
x1 := simplify(rhs(env[1][1])):
y1 := simplify(rhs(env[1][2])):
# parametrization of 2nd component - 2nd envelope
x2 := simplify(rhs(env[2][1])):
y2 := simplify(rhs(env[2][2])):
comp1 := plot([x1, y1, t = 0 .. 2*Pi], color = green):
comp2 := plot([x2, y2, t = 0 .. 2*Pi], color = red):
# the original deltoid
delt := plot([2*cos(t) + cos(2*t), 2*sin(t) - sin(2*t), t = 0 .. 2*Pi],
color = navy, thickness = 3):
display(delt, comp1, comp2):
circles := animate(implicitplot, [F = 0, x = -4 .. 5, y = -4 .. 4], t = 0 .. 2*Pi,
frames = 50):
display(axes, delt, comp1, comp2, circles):
# this display shows the circles when the center moves along the deltoid

```

In order to run the animation, right click on the picture and use the dedicated buttons which appear on the screen.

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