PROBLEM CORNER

Proposed by

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To become a mathematics teacher in a public High School in Spain it is not essential to have a Mathematics B.Sc. degree (it can be a degree on a different scientific discipline) but you do have to achieve, first, a Master degree focusing on relevant pedagogical aspects for pre-service teacher training. Then, you have to pass some public exam (usually called "Oposiciones" by Spanish people), that usually takes place every two years, and it is organized and implemented in each of the Spanish regions (as the development of national education policies is a competence of the different regional authorities, like it happens with the States in the U.S.A.). Obviously, getting prepared and passing the exam is a very relevant task for thousands of young graduate students, that aim to start an professional career as mathematics teachers.

Roughly speaking, the exam includes three parts, where the candidate is asked to:

- give a lecture on the mathematics underlying a certain subject chosen by the examination jury from a publicly announced list, related to the Mathematics Secondary Education Curriculum (e.g. "Limit of functions, notion of continuity, Bolzano's Theorem"),
- to solve some problems, proposed by the jury, of a level corresponding to the first two years of the Bachelor Degree in Mathematics in any of the Spanish Universities, and
- to present and defend a written project dealing with their personal perception of the didactical issues involved in the teaching of mathematics courses in the Secondary Education classroom.

Let us remark that no technological tool (calculator, tablet, laptop, smarthphone, ...) is allowed during the exam, for addressing the proposed problems.

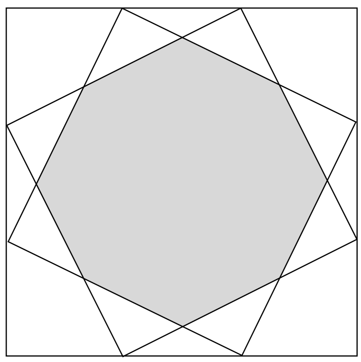
The last of these public "Oposiciones" exams was carried out in June 2021, and among the problems proposed in that call we have chosen two that we find especially attractive, due to their geometric content, and because they allow us to present more than one solution and to discuss some subtle details.

Bearing in mind the restrictions of the "Oposiciones" exam, we have not consider the use of CAS or DGS tools to solve these problems. Yet, we would highly appreciate to receive solutions from the readers of the eJMT Problem Corner that emphasize the potential role of CAS/DGS to solve (fully or partially), explore or extend such problems.

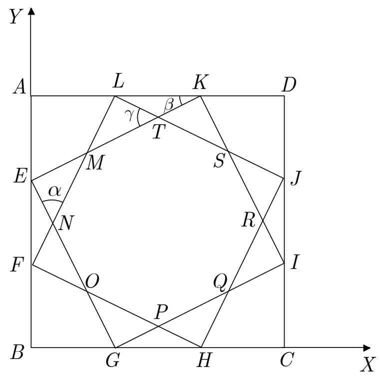
We encourage the readers of the eJMT Problem Corner to attempt solving them, as so many young Spanish candidates to become mathematics teachers ought to do!!

Problem 1

The shaded area of the figure measures 100 cm^2. The two overlapping smaller squares are equal. The side of the larger square is divided into three segments of equal length by the vertices of the smaller squares. Calculate the area of the largest square.



First solution. Let us find firstly the relationship between the lengths of the sides of the right triangle $\triangle NEM$ in the figure.



To that end we will employ coordinates with respect to the orthonormal reference system

$$\mathcal{R} := \{B;\ \vec{u} := \overrightarrow{BG},\ \vec{v} := \overrightarrow{BF}\}$$

in the euclidean plane. Thus, $B = (0,0)$, $F = (0,1)$ and $L = (1,3)$, so $\overrightarrow{FL} = (1,2)$. In addition, $E = (0,2)$ and $G = (1,0)$, thus $\overrightarrow{GE} = (-1,2)$. Therefore, the dot product of these vectors is

$$\overrightarrow{FL} \cdot \overrightarrow{GE} = 1 \cdot (-1) + 2 \cdot 2 = 3.$$

Let $\alpha := \measuredangle MNE$. The above dot product can be calculated as

$$3 = \overrightarrow{FL} \cdot \overrightarrow{GE} = FL \cdot GE \cdot \cos\alpha = 5\cos\alpha, \quad \text{that is,} \quad \cos\alpha = \frac{3}{5}.$$

Let $\ell := MN/5$. Since $\triangle NEM$ is a right triangle,

$$\frac{3}{5} = \cos\alpha = \frac{EN}{MN} = \frac{EN}{5\ell} \quad \text{and} \quad \frac{4}{5} = \sin\alpha = \frac{EM}{MN} = \frac{EM}{5\ell},$$

so $EN = 3\ell$ and $EM = 4\ell$. We calculate next FL in terms of ℓ.

Since angles opposite by the vertex have the same measure it follows that the triangles

$$\triangle NEM, \quad \triangle NFO, \quad \triangle PGO, \quad \triangle PHQ, \quad \triangle RIQ, \quad \triangle RJS, \quad \triangle TKT \quad \text{and} \quad \triangle TLM$$

are equal. In particular, the equality $\triangle NEM = \triangle TLM$ implies $LM = EM = 4\ell$ and the equality $\triangle NEM = \triangle NFO$ implies $FN = EN = 3\ell$. Therefore

$$FL = FN + NM + ML = 3\ell + 5\ell + 4\ell = 12\ell.$$

Let us compute now the value of ℓ^2. Let $\mathcal{K}$ denote the shaded octagon in the figure and let $\mathcal{Q}$ be the square whose vertices are F, H, J y L. Then,

$$\begin{aligned}
100 &= \text{area}\,(\mathcal{K}) \\
&= \text{area}\,(\mathcal{Q}) - [\text{area}\,(\triangle NFO) + \text{area}\,(\triangle PHQ) + \text{area}\,(\triangle SJR) + \text{area}\,(\triangle TLM)] \\
&= \text{area}(\mathcal{Q}) - 4 \cdot \text{area}(\triangle NEM) = FL^2 - 4 \cdot \left(\frac{EN \cdot EM}{2}\right) \\
&= (12\ell)^2 - 2 \cdot 3\ell \cdot 4\ell = 144\ell^2 - 24\ell^2 = 120\ell^2,
\end{aligned}$$

hence $\ell^2 = 5/6$ cm^2. Therefore,

$$FL^2 = \text{area}\,(\mathcal{Q}) = 144\ell^2 = 120.$$

Finally, to compute the area of the square $\mathcal{C}$ whose vertices are A, B, C and D we notice that área $(\mathcal{C}) = AB^2$ and

$$AF = \frac{2 \cdot AB}{3}, \quad AL = \frac{AD}{3} = \frac{AB}{3}.$$

Hence,

$$120 = FL^2 = AF^2 + AL^2 = \frac{5}{9} \cdot AB^2 \quad \text{and}$$

$$\text{area}\,(\mathcal{C}) = AB^2 = \frac{120 \cdot 9}{5} = 216 \text{ cm}^2.$$

Comment. Notice that the lengths of the sides of the octagon $\mathcal{K}$ coincide, but this is not so with the measures of its angles. Thus, $\mathcal{K}$ is not a regular octagon!

Second solution. Let $x := AB$ be the length of the sides of the largest square and let $y := EK$ be the length of the sides of the smallest squares. Applying Pythagoras theorem in the triangle $\triangle EAK$ we have

$$y^2 = EK^2 = EA^2 + AK^2 = \left(\frac{x}{3}\right)^2 + \left(\frac{2x}{3}\right)^2 = \frac{5x^2}{9}. \tag{1}$$

Let us denote, as in the figure, $\beta := \measuredangle LKT$ and $\gamma := \measuredangle MTL$. By the symmetry, $KT = LT$, so $\triangle KTL$ is an isosceles triangle. Thus

$$\gamma = \measuredangle MTL = \pi - \measuredangle KTL = \measuredangle LKT + \measuredangle TLK = 2\measuredangle LKT = 2\beta.$$

Let us compute next the sinus and cosine of γ. Note first that

$$\tan\beta = \frac{AE}{AK} = \frac{1}{2} \quad \text{and}$$

$$\sin\gamma = \sin 2\beta = 2\sin\beta\cos\beta = \frac{2\sin\beta\cos\beta}{\cos^2\beta + \sin^2\beta} = \frac{2\tan\beta}{1+\tan^2\beta} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}.$$

In addition,

$$\cos\gamma = \cos 2\beta = \cos^2\beta - \sin^2\beta = \frac{\cos^2\beta - \sin^2\beta}{\cos^2\beta + \sin^2\beta} = \frac{1-\tan^2\beta}{1+\tan^2\beta} = \frac{1-\frac{1}{4}}{1+\frac{1}{4}} = \frac{3}{5}.$$

Let us denote $h := MT$. Then,

$$ML = MT \cdot \sin\gamma = \frac{4h}{5} \quad \text{and} \quad LT = MT \cdot \cos\gamma = \frac{3h}{5}. \tag{2}$$

Therefore,

$$y = EK = FL = FN + NM + ML = LT + MT + ML = \frac{3h}{5} + h + \frac{4h}{5} = \frac{12h}{5},$$

that is, $h = \frac{5y}{12}$ and, using (2),

$$ML = \frac{4h}{5} = \frac{y}{3} \quad \text{and} \quad LT = \frac{3h}{5} = \frac{y}{4}, \quad \text{and so} \quad \text{area}\,(\triangle TLM) = \frac{ML \cdot LT}{2} = \frac{y^2}{24}.$$

As in the first solution, let $\mathcal{K}$ be the shaded octagon in the figure and let $\mathcal{Q}$ be the square whose vertices are F, H, J and L. Then, by (1),

$$100 = \text{area}\,(\mathcal{K}) = \text{area}\,(\mathcal{Q}) - [\text{area}\,(\triangle NFO) + \text{area}\,(\triangle PHQ) + \text{area}\,(\triangle SJR) + \text{area}\,(\triangle TLM)],$$

that is,

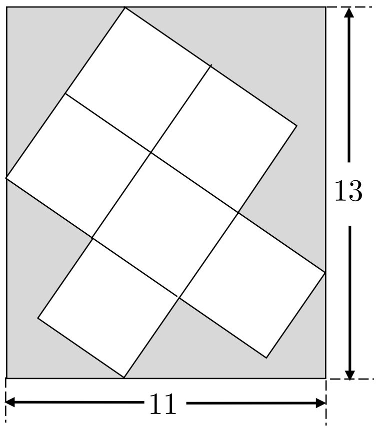
$$100 = y^2 - \frac{4y^2}{24} = \frac{5y^2}{6} = \frac{5}{6} \cdot \frac{5x^2}{9} = \frac{25x^2}{54},$$

and the area of the largest square $\mathcal{C}$ is

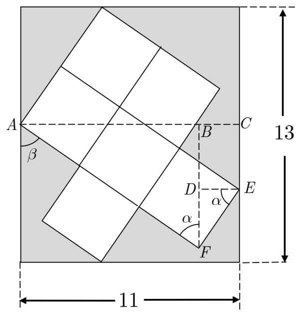
$$\text{area}\,(\mathcal{C}) = x^2 = 4 \cdot 54 = 216 \text{ cm}^2.$$

Problem 2

The rectangle in the figure contains six equal squares arranged as indicated. Determine the side of any of these squares.



Solution. We present two solutions of this problem. We denote ℓ the length of the side of each one of the squares occurring in the statement.



First solution. With the notations in the figure let r_1 be the line parallel to the horizontal sides of the rectangle in the statement and passing through the point A. Let r_2 be the line orthogonal to r_1 passing through F, and let r_3 be the parallel to r_1 passing through E. Let us define $B := r_1 \cap r_2$ and $D := r_2 \cap r_3$. Note that

$$\frac{\pi}{2} - \measuredangle BFA = \frac{\pi}{2} - \measuredangle DEF, \quad \text{so} \quad \measuredangle BFA = \measuredangle DEF := \alpha,$$

Since $\triangle ABF$ is a right triangle we have

$$\beta = \measuredangle FAB = \frac{\pi}{2} - \measuredangle BFA = \frac{\pi}{2} - \alpha.$$

Thus, focussing in the triangle $\triangle ABF$ we get

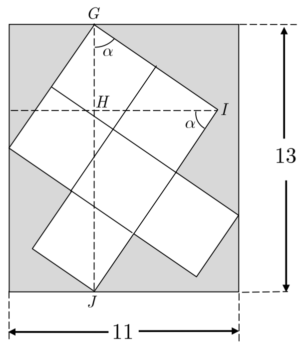
$$\sin\alpha = \frac{AB}{AF} = \frac{AB}{3\ell}, \quad \text{that is,} \quad AB = 3\ell \sin\alpha. \tag{3}$$

On the other hand $\triangle EDF$ is a right triangle too, so

$$\cos\alpha = \frac{DE}{EF} = \frac{BC}{\ell}, \quad \text{that is,} \quad BC = \ell\cos\alpha. \tag{4}$$

Adding the equalities (3) y (4) it follows

$$11 = AB + BC = AB + DE = 3\ell \sin \alpha + \ell \cos \alpha. \tag{5}$$



Denote G, I and J the vertices of the small squares as in the figure above. Let r_4 be the line parallel to r_1 passing through I and let r_5 be the line orthogonal to r_4 passing through G. Denote $H := r_4 \cap r_5$ and note that $\measuredangle HGI = \alpha$ since the line ℓ_{GH} passing through G and H is parallel to the line ℓ_{DF} passing through D and F and the line ℓ_{GI} passing through G and I is parallel to the line ℓ_{AF} passing through A and F. Thus, since $\triangle GHI$ is a right triangle,

$$\cos \alpha = \frac{GH}{GI} = \frac{GH}{2\ell}, \quad \text{that is,} \quad GH = 2\ell \cos \alpha. \tag{6}$$

On the other hand, let r_6 be the line orthogonal to the horizontal sides of the rectangle in the statement passing through J and let $L := r_6 \cap r_4$. We will see in the second solution that L and H coincide, but right now we do not know this is so. This is why we do not draw the point L. In the right triangle $\triangle ILJ$ we have

$$\sin \alpha = \frac{LJ}{IJ} = \frac{\operatorname{dist}(J, r_5)}{3\ell}, \quad \text{that is,} \quad \operatorname{dist}(J, r_5) = 3\ell \sin \alpha. \tag{7}$$

From equalities (6) y (7) it follows

$$13 = \operatorname{dist}(J, r_4) + \operatorname{dist}(G, r_4) = 3\ell \sin \alpha + GH = 3\ell \sin \alpha + 2\ell \cos \alpha. \tag{8}$$

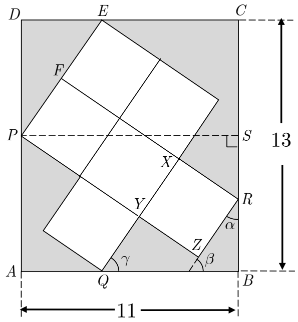
Thus, using (5) and (8) we get

$$11 = \ell(3 \sin \alpha + \cos \alpha) \quad \text{and} \quad 13 = \ell(3 \sin \alpha + 2 \cos \alpha). \tag{9}$$

Substracting both equalities, $2 = \ell \cos \alpha$, thus $11 = 3\ell \sin \alpha + 2$, that is, $3 = \ell \sin \alpha$. Finally,

$$1 = \cos^2 \alpha + \sin^2 \alpha = \frac{4+9}{\ell^2} = \frac{13}{\ell^2}, \quad \text{thus} \quad \ell = \sqrt{13}.$$

Second solution. We denote ℓ the length of the side of each one of the squares occurring in the statement but, to avoid confusion, we introduce new notations, independent of the ones employed in the first solution.



The triangle in the lower right corner is a right triangle, so $\alpha+\beta=\pi/2$. In addition we have $\gamma=\beta=\pi/2-\alpha$ because the measures of the angles formed by the intersections of two parallel lines by a third one coincide. Let S be the intersection point of the line joining B and C with the line orthogonal to this last passing through the point P. Note that

$$\measuredangle SRP+\measuredangle PRZ+\measuredangle ZRB=\measuredangle SRB=\pi.$$

Thus $u:=\measuredangle PRZ$ satisfies $\measuredangle SRP+u+\alpha=\pi$, and so

$$\frac{11}{PR}=\frac{PS}{PR}=\sin\measuredangle SRP=\sin(\pi-u-\alpha)=\sin(u+\alpha).$$

Let us compute PR in terms of ℓ. As $PZ=3\ell$ and $ZR=\ell$, by Pythagoras Theorem it follows,

$$PR=\sqrt{PZ^2+ZR^2}=\sqrt{10\ell^2}=\sqrt{10}\ell,$$

which implies

$$\sin(u+\alpha)=\frac{11}{PR}=\frac{11}{\sqrt{10}\ell}. \tag{10}$$

Since $\triangle ETQ$ is a right triangle with $ET=2\ell$ and $TQ=3\ell$ we get

$$EQ=\sqrt{ET^2+TQ^2}=\sqrt{13}\ell \ \text{ and so, } \ \sin\measuredangle BQE=\frac{\operatorname{dist}(AB,\,CD)}{EQ}=\frac{13}{\sqrt{13}\ell}. \tag{11}$$

Let $v:=\measuredangle TQE$. Then

$$\measuredangle BQE=\measuredangle BQT+\measuredangle TQE=\gamma+v=\pi/2-\alpha+v$$

and the equality (11) can be rewritten

$$\cos(v-\alpha)=\cos(\alpha-v)=\sin(\pi/2-\alpha+v)=\sin\measuredangle BQE=\frac{13}{\sqrt{13}\ell}. \tag{12}$$

On the other hand,

$$\sin u=\frac{PZ}{PR}=\frac{3\ell}{\sqrt{10}\ell}=\frac{3}{\sqrt{10}}, \quad \text{so} \quad \cos u=\frac{1}{\sqrt{10}},$$

whereas

$$\sin v = \frac{ET}{EQ} = \frac{2\ell}{\sqrt{13}\ell} = \frac{2}{\sqrt{13}}, \quad \text{which implies} \quad \cos v = \frac{3}{\sqrt{13}}.$$

Equalities (10) y (12) are written now as

$$\frac{11}{\sqrt{10}\ell} = \sin(u+\alpha) = \sin u \cos\alpha + \cos u \sin\alpha = \frac{1}{\sqrt{10}} \cdot (3\cos\alpha + \sin\alpha),$$

$$\frac{13}{\sqrt{13}\ell} = \cos(v-\alpha) = \cos v \cos\alpha + \sin v \sin\alpha = \frac{1}{\sqrt{13}} \cdot (3\cos\alpha + 2\sin\alpha).$$

Therefore,

$$11 = \ell(3\cos\alpha + \sin\alpha) \quad \text{and} \quad 13 = \ell(3\cos\alpha + 2\sin\alpha).$$

Substracting both equalities, $2 = \ell\sin\alpha$, thus $11 = 3\ell\cos\alpha + 2$, that is, $3 = \ell\cos\alpha$. Finally,

$$1 = \cos^2\alpha + \sin^2\alpha = \frac{9+4}{\ell^2} = \frac{13}{\ell^2} \quad \text{thus} \quad \ell = \sqrt{13}.$$

Comment. If the reader takes square and bevel, he shall verify that, at least graphically, it seems clear that the lines ℓ_{EQ} and ℓ_{AB} passing through E and Q and through A and B, respectively, are orthogonal. We have been unable to prove this fact a fortiori, although it follows easily once the length $\ell = \sqrt{13}$ is known. Indeed,

$$\sin \measuredangle BQE = \frac{13}{\sqrt{13}\ell} = 1, \quad \text{hence} \quad \measuredangle BQE = \frac{\pi}{2}, \quad \text{and so} \quad \ell_{EQ} \perp \ell_{AB}.$$

Acknowledgements

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References

[1] José Manuel Gamboa, Francisco Baena, Braulio de Diego. *Problemas de Oposiciones. Matemáticas.* Tomo 10. Editorial Deimos. (2021) ISBN:= 978-84-86379-98-8