Analytic, Geometric, and Numeric Analysis of the Shrinking Circle and Sphere Problems

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Abstract

The Shrinking Circle Problem is an example of a simple-to-state geometry problem that is visually appealing yet quite challenging to solve. A combination of geometry and analysis is used to completely solve the general problem in the plane, and its extension to three dimensions: the Shrinking Sphere Problem. We show why traditional numerical attempts to answer even the simplest problem is futile. The original problem was generalized based on visual evidence produced by dynamic geometry software. Only with this insight was it possible to utilize symbolic computation tools to put together the complete proofs. All supplemental materials that accompany this paper can be found online at either of the folowing URLs:
http://www.math.sc.edu/~meade/eJMT-Shrink/
http://www.radford.edu/~scorwin/eJMT/Content/Papers/v1n1p4.

1 Introduction

The following problem appears, with slightly different notation, as an exercise in [Stewart, 2007, p. 45, Exercise 56].

Problem 1 (The Original Shrinking Circle Problem) *Figure 1 shows a fixed circle C with equation $(x-1)^2 + y^2 = 1$ and a shrinking circle C_r with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_r shrinks, that is, as $r \to 0^+$?*

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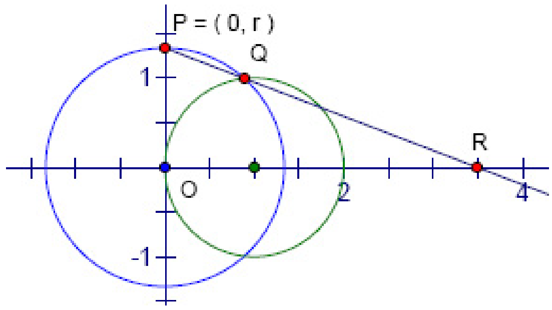


Figure 1: The Original Shrinking Circle Problem. What happens to R as $r \to 0^+$? (Plot produced with [GSketchpad], [1].)

It is clear that the slope of the segment PR is negative and increasing towards zero as the point P approaches the origin. One reasonable conjecture is that the point R diverges towards infinity. In fact, in Section 2, it will be shown that the limit does exist, and is 4. The discussion in Section 2 is based on an analytic proof of this fact. It also includes the first indication of some of the subtle analytic features that will be so important in the remainder of the paper.

The Generalized Shrinking Circle Problem is obtained by replacing the fixed circle C with a fixed general curve (also called C). A geometric proof is given in [Kreczner, 1995] and an analytic proof has recently been given in [Yang, 2006]. Regardless of the techniques used in the proof, the key idea is that the behavior of this limit is completely determined by the osculating circle of the fixed curve at the origin.

The discussions in Sections 2 and 3 lead in a natural way to corresponding problems in three dimensions. The Original Shrinking Sphere Problem is obtained in Section 4 by replacing C with S, the fixed unit sphere centered at $(1, 0, 0)$, and replacing C_r with S_r, a shrinking sphere with radius r and center at the origin.

The Generalized Shrinking Sphere Problem is obtained by replacing the fixed sphere S by any fixed surface in $\mathbf{R}^3$. A complete characterization of the limit in these cases is given in Section 5. Not surprisingly, this result depends critically on an osculating circle of the fixed surface at the origin.

In both two and three dimensions, the results are confirmed for several special geometries. Also, each general result is restated in a form that does not make any reference to a global coordinate system.

Section 6 revisits the key analytic results involved in the proofs of Theorem 2 and Theorem 5. This discussion includes specific examples that illustrate the difficulties involved in trying to gain insight into these problems using only numerical or graphical evidence. References to downloadable electronic materials related to the analysis of these problems are provided throughout the paper. All of these materials are collected on a single webpage at the URL: `http://www.math.sc.edu/~meade/eJMT-Shrink/`.

2 The Original Shrinking Circle Problem

The original problem as it appears in [Stewart, 2007] can be solved by brute force. The equation of C_r is $x^2 + y^2 = r^2$ and the point Q where C and C_r intersect in the first quadrant is found to be $\left(\frac{r^2}{2}, \frac{r}{2}\sqrt{4 - r^2}\right)$. The x-coordinate of point R where the line PQ intersects the x-axis is

r	d=12	d=10	d=8	d=6	d=4
1.0	3.732	3.732	3.732	3.732	3.731
0.5	3.936	3.936	3.936	3.936	3.906
0.1	3.997	3.997	3.997	4.000	3.333
0.05	3.999	3.999	3.999	3.968	2.500
0.01	3.999	4.000	4.000	3.333	NaN
0.005	3.999	4.000	3.968	2.500	NaN
0.001	4.000	4.000	3.333	NaN	NaN
0.0005	4.000	3.968	2.500	NaN	NaN
0.0001	4.000	3.333	NaN	NaN	NaN
0.00005	3.968	2.500	NaN	NaN	NaN
0.00001	3.333	NaN	NaN	NaN	NaN

Table 1: Numerical approximation to x_R for moderately small values of r and different numbers of significant digits. [All calculations performed with [Maple]. [2].]

$x_R = \frac{-r}{m}$ where $m = \frac{r-\frac{r}{2}\sqrt{4-r^2}}{0-\frac{r^2}{2}}$. Thus, $x_R = \frac{-r}{m} = \frac{r^2}{2-\sqrt{4-r^2}}$.

As the circle of C_r shrinks to the point at the origin, the expression for x_R is indeterminate of form $\frac{0}{0}$.

The indeterminate form of this limit interferes wit- attempts to evaluate it numerically. Table 1 shows numerical values of x_R for moderately small values of r when the floating point arithmetic is performed using d significant digits.[1] While the precise value of r where the computed solutions begin to deteriorate decreases as the number of digits used increases, it is clear that this behavior will be observed for any fixed finite number of significant digits.[2]

The values in Table 1 do suggest that the limit, should it exist, is most likely to have a value of 4. That this is the correct value for this limit is decided once and for all with one application of l'Hôpital's Rule:

$$\lim_{r\to 0^+} x_R = \lim_{r\to 0^+} \frac{r^2}{2-\sqrt{4-r^2}} = \lim_{r\to 0^+} \frac{2r}{\frac{r}{\sqrt{4-r^2}}} = \lim_{r\to 0^+} 2\sqrt{4-r^2} = 4.$$

where l'Hôpital's Rule is used one time when the limit has indeterminate form $\frac{0}{0}$.

The first step to solving the Original Shrinking Circle Problem is to have a visual understanding of the construction of the point R. While this visualization can be accomplished in a CAS such as Maple [Maple] or *Mathematica* [Mathematica], the construction is much simpler and the resulting animation is better when a dynamic geometry tool such as Cabri 3D [Cabri3D], Geometer's Sketchpad [GSketchpad], and Geometry Expressions [GExpressions] is used.

[1] Maple's default is to use 10 significant digits in all floating-point calculations; many TI calculators use 12 significant digits.

[2] This is also a good illustration that using d significant digits in all calculations does not guarantee that the computed value is accurate to d digits.

3 The Generalized Shrinking Circle Problem

Consider the following extension of the Original Shrinking Circle Problem:

Problem 2 (The Generalized Shrinking Circle Problem) *Let C be a fixed curve and let C_r be the circle with center at the origin and radius r. P is the point $(0, r)$, Q is the upper point of intersection of C and C_r, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_r shrinks, that is, as $r \to 0^+$?*

A first observation is that if C does not include the point at the origin, then the points R are not defined for all values of r in a one-sided neighborhood of 0.

An explicit general formula for the coordinates of R is not going to be readily available, except in special cases. This means that neither of the approaches used to analyze the Original Shrinking Circle Problem (calculus and numerical) will be useful for the general case.

An important step towards analyzing the Generalized Shrinking Circle Problem is to consider the case when the fixed curve is any circle that passes through the origin.

Lemma 1 *Let C be the circle with center (a, b) that includes the origin: $(x-a)^2 + (y-b)^2 = a^2 + b^2$. Let C_r, P, Q, and R be defined as in the Generalized Shrinking Circle Problem. Then*

$$\lim_{r \to 0^+} R = \begin{cases} (4a, 0) & \text{if } b = 0 \\ (0, 0) & \text{otherwise} \end{cases}$$

Proof. [3]

The full step-by-step derivation of this result has been done in a Maple worksheet [3]. The highlights of that development are given here:

For these circles it is still possible to obtain explicit formula for the points Q and R:

$$Q = \left(\frac{r}{2} \frac{ar - b\,\mathrm{sgn}(a)\sqrt{4(a^2+b^2) - r^2}}{a^2+b^2}, \frac{r}{2} \frac{br + |a|\sqrt{4(a^2+b^2) - r^2}}{a^2+b^2} \right)$$

$$R = \left(\frac{r\left(ra - b\,\mathrm{sgn}(a)\sqrt{4(a^2+b^2) - r^2}\right)}{2(a^2+b^2) - br - |a|\sqrt{4(a^2+b^2) - r^2}}, 0 \right)$$

where sgn is the "sign" function ($\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(0) = 0$, and $\mathrm{sgn}(x) = -1$ if $x < 0$).

For $b = 0$, the expression for the x-component of R is $x_R = \frac{ar^2}{2a^2 - |a|\sqrt{4z^2 - r^2}}$. A straightforward calculation shows that $x_R \to 4a$ as $r \to 0^+$. When $b \neq 0$, R does not have an indeterminate form and $R \to 0$ as $r \to 0^+$. ■

Let $\mathcal{C}$ denote the osculating circle of C at the origin. This means $\mathcal{C}$ intersects C at the origin, is tangent to C at the origin, and has the same curvature as C at the origin. Another way to express this is to say that C and $\mathcal{C}$ have 3-point contact at the origin.

Suppose C has curvature κ at the origin. The osculating circle $\mathcal{C}$ will be the circle with radius $\rho = \frac{1}{\kappa}$ whose center (a, b) is on the normal line to C at the origin. Thus, $a^2 + b^2 = \rho^2$.

[3] A purely geometric proof of Lemma 1 has been suggested to the authors by Tom Banchoff. This proof will be the subject of a separate paper, yet to be written.

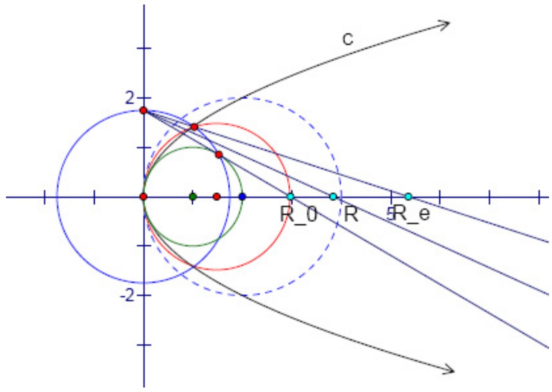


Figure 2: Construction for the proof of Theorem 2. (Plot produced with [GSketchpad], [4].)

Theorem 2 *Let C be a curve in the plane that includes the origin and is twice continuously differentiable at the origin. Define C_r, P, Q, and R as in the Generalized Shrinking Circle Problem. If the curvature at the origin, κ, is positive, the osculating circle of C at the origin has radius $\rho = \frac{1}{\kappa}$ and center (a, b) where $a^2 + b^2 = \rho^2$. Then,*

$$\lim_{r \to 0^+} R = \begin{cases} (4\rho, 0) & \textit{if } b = 0 \\ (0, 0) & \textit{otherwise.} \end{cases}$$

Proof. This proof is based on the one found in [Kreczner, 1995].

The osculating circle of C at the origin, $\mathcal{C}$, has radius $\rho = \frac{1}{\kappa}$ and center (a, b) where $a^2 + b^2 = \rho^2$.

Case 1: Assume $b = 0$ and $a = \rho$. This means that, near the origin, the graph of C is concave down and in the first quadrant. Let $\mathcal{C}_\epsilon$ denote the circle with radius $\rho + \frac{\epsilon}{4}$ and center $(\rho + \frac{\epsilon}{4}, 0)$. Note that $\mathcal{C}_0 = \mathcal{C}$, the osculating circle of C at the origin. Define Q_ϵ to be the intersection of C_r and $\mathcal{C}_\epsilon$ in the first quadrant and R_ϵ to be the intersection of the line through P and Q_ϵ with the (positive) x-axis. Even though $\mathcal{C}_0 = \mathcal{C}$, Q_0 and R_0 do not necessarily coincide with Q and R, respectively.

Let $\epsilon > 0$ be given. For $r > 0$ but sufficiently close to 0, the points R_0, R, and R_ϵ appear in order from left to right on the positive x-axis (see Figure 2). By Lemma 1, $R_0 \to (4\rho, 0)$ and $R_\epsilon \to (4\rho + \epsilon, 0)$ as $r \to 0^+$, for all $\epsilon > 0$.

Now, let $\epsilon \to 0^+$, and conclude that $\lim_{r \to 0^+} R = (4\rho, 0)$.

Case 2: When $b = 0$ and $a = -\rho$, the graph of C is concave down and in the second quadrant. Reflection across the y-axis transforms the problem into Case 1.

Case 3: A similar approach can be used when $b \neq 0$. The difference here is that R_0 and R_ϵ both approach $(0, 0)$ as $r \to 0^+$. This case also includes the situations where $\kappa = 0$. ■

3.1 Other Geometries

Theorem 2 can be verified for other curves besides a circle. Table 2 shows the results for three general classes of curves. The ellipse is a natural generaliation from the circle. The lines through the origin have $\kappa = 0$ and hence an osculating circle does not exist. The parabola is the example considered in [Kreczner, 1995], for which *Mathematica* was reportedly unable to correctly evaluate. The details for each case are provided in supplemental Maple worksheets.

Curve	C	κ	$\lim_{r\to0^+} R$	Reference
ellipse	$\left(\frac{x-a}{a}\right)^2+y^2=1$	a	$\frac{4}{a}$	[5]
line	$y=ax$	0	0	[6]
parabola	$y^2=ax$	$\frac{2}{a}$	$2a$	[7]

Table 2: Summary of results for non-circular fixed curves. Each worksheet contains the full derivation for the specfic geometry, including an animation of the limiting process.

3.2 Coordinate-Free Version

The solution to the Generalized Shrinking Circle Problem can be restated without any mention to a global coordinate system. The analysis used in the proof of Theorem 2 does depend on a particular choice for the coordinate system. This is not the only way to give the proof. Putting aside some of the complicated symbolic representations for Q and R, the essential ideas are all geometric. To emphasize the geometric nature of the problem, and its solution, we restate the result without any use of coordinates.

Theorem 3 *Let $\mathcal{O}$ be a point on a curve C in the plane where the osculating circle to C at $\mathcal{O}$ exists. Let $\mathbf{T}$ and $\mathbf{N}$ be the unit tangent and normal vectors to C at $\mathcal{O}$, respectively. Let κ be the curvature of C at $\mathcal{O}$. ($\mathbf{N}$ is oriented so that $\mathcal{O}+\frac{1}{\kappa}\mathbf{N}$ is the center of the osculating circle to C at $\mathcal{O}$.) For any $r>0$, define*

- *C_r to be the circle with radius r centered at $\mathcal{O}$,*
- *$P=\mathcal{O}+r\mathbf{T}$, the point at the top of C_r,*
- *Q to be the intersection of C and C_r, and*
- *R to be the point on the line through P and Q such that $\overline{\mathcal{O}R}$ is parallel with $\mathbf{N}$*

Then, as r decreases to 0, R converges to the point $R_0=\mathcal{O}+\frac{4}{\kappa}\mathbf{N}$.

This formulation of the solution to the Generalized Shrinking Circle Problem will be used in the analysis of the corresponding three-dimensional problem, the Shrinking Sphere Problem.

4 The Simplest Shrinking Sphere Problem

The success with the two-dimensional problems piques interest in the corresponding three-dimensional problems.

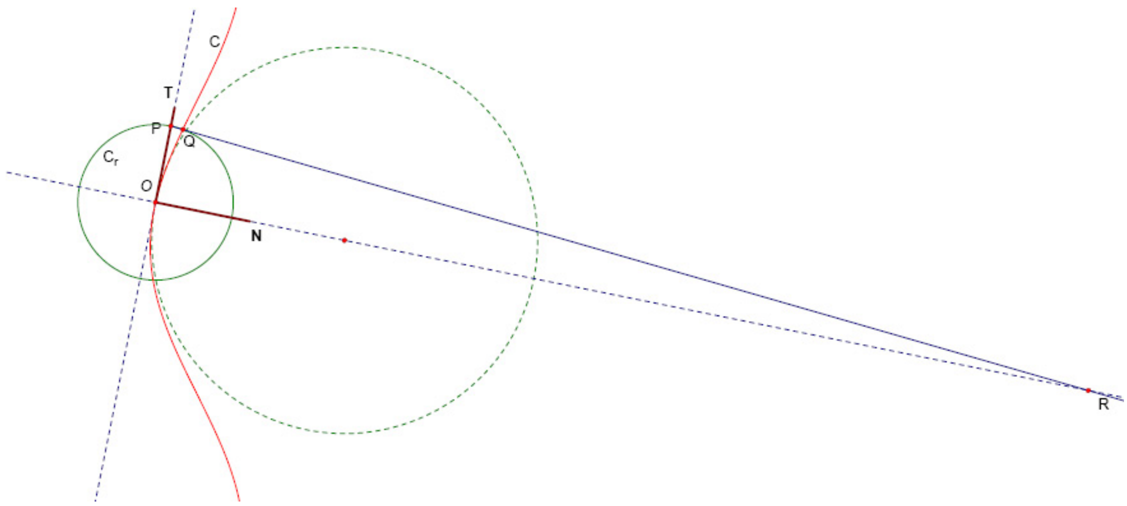


Figure 3: The coordinate-free geometry of Theorem 3. (Plot produced with [GSketchpad], [8].)

Problem 3 (The Simplest Shrinking Sphere Problem) *Let S be the unit sphere centered at $(0,1,0)$ and let S_r denote the sphere centered at the origin with radius r. Let P be the point at the top of S_r, $(0,0,r)$, let Q be the intersection of spheres S and S_r, and let R be the projection of Q onto the $x-y$ plane. What is $\lim_{r\to 0^+} R$?*

The intersection, Q, between S_r and S is a circle. In fact, it is the circle perpendicular to the y-axis with radius $\frac{r}{2}\sqrt{4-r^2}$ and center $(0,\frac{r^2}{2},0)$. The projection of the point $(0,0,r)$ through the highest point on Q to the $z=0$ plane corresponds to the Original Shrinking Circle Problem discussed in Section 2; this point must converge to $(4,0,0)$ as $r\to 0^+$.

The projection of P through each point of Q onto the $z=0$ plane forms a new curve R. In this case the points on R satisfy $x^2+(y-2)^2=4-r^2$, the circle in the $z=0$ plane with center $(0,2,0)$ with radius $\sqrt{4-r^2}$. As S_r shrinks to the single point at the origin, these curves converge to the circle $x^2+(y-2)^2=4$.

The analysis of the Simplest Shrinking Sphere Problem is more subtle than it first appears. To illustrate, for each point (x,y,z) define the angle $\theta=\arctan\frac{x}{z}\in(-\pi,\pi]$ with the signs of x and z used to determine the appropriate quadrant for the angle. Note that θ is the angle made with the positive z-axis measured in the plane parallel to the $y=0$ plane. Let $\theta\in(-\pi,\pi]$ be given. Consider the cross-section of the construction with S and S_r obtained by slicing with the plane that makes an angle θ with the positive z-axis. Note that the y-axis is contained in this cross-section, hence the centers of both S and S_r are included. The slice of S_r is a circle with radius r centered at the origin. The slice of S is a fixed circle with the same center and radius as S. Their intersection is a point on Q and its projection onto the $z=0$ plane, from $P=(0,0,r)$, is a point on R; denote these last two points as Q_θ and R_θ, respectively. When $\theta=0$ (and $\theta=\pi$) the point P is also in this plane. This is precisely the setup for the Shrinking Circle Theorem; in this case the point R_θ approaches the point $(\frac{4}{\kappa},0)$ as $r\to 0^+$. For all other values of θ the point P is not in the plane. This is similar to Lemma 1 when $b\neq 0$. The general observation from this is that if the points Q_θ and R_θ do not lie in the plane that includes the centers of S and S_r and the point P, then these points approach the origin as S_r shrinks to the point at the origin.

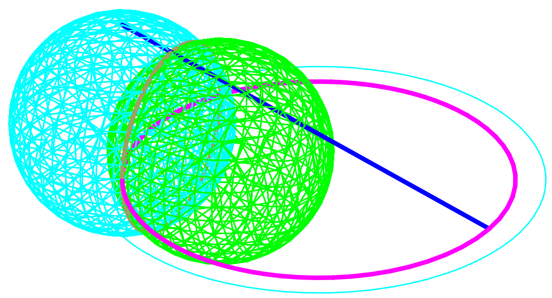


Figure 4: The Simplest Shrinking Sphere Problem. (Plot created with [Maple], [9].)

This observation will be confirmed in the next section during the proof of the solution to the Generalized Shrinking Sphere Problem.

5 The Generalized Shrinking Sphere Problem

We are now prepared to extend the results of the Simplest Shrinking Sphere Problem to surfaces S that are not spheres.

Problem 4 (The Generalized Shrinking Sphere Problem) *Let S be a fixed surface and let S_r be the sphere with center at the origin and radius r. P is the point $(0, 0, r)$, Q is the curve of intersection between S and S_r, and R is the projection from P through Q onto the $z = 0$ plane. Describe $\lim_{r\to 0^+} R$.*

The general ideas used in the Generalized Shrinking Circle Problem can be extended to three dimensions. This suggests approximating the surface S with an appropriate sphere.

Lemma 4 *Let S be the sphere centered at $(0, a, b)$ that includes the origin: $x^2+(y-a)^2+(z-b)^2 = a^2+b^2$. Let S_r, P, Q, and R be defined as in the Generalized Shrinking Sphere Problem. Then,*

$$\lim_{r\to 0^+} R = \begin{cases} x^2+(y-2a)^2 = 4a^2 & \text{if } a \neq 0 \text{ and } b = 0 \\ (0,0) & \text{otherwise} \end{cases}$$

Before proving this result, take a minute to notice how Lemma 4 is a natural extension of Lemma 1. In both cases the interesting cases are when *(i)* the radial vector from the center of the fixed object to the origin and the vector from the origin to the point P are perpendicular and *(ii)* the point P and the centers of S_r and S are coplanar. In these cases, $\lim_{r\to 0^+} R$ is the set of points $\mathbf{x}$ that satisfies $\|\mathbf{x}-\mathbf{a}\| = 2\|\mathbf{a}\|$ where $\|\cdot\|$ is the appropriate Euclidean norm.

Proof. [4]

[4]Additional justification of the steps in this proof are provided in a supplemental Maple worksheet [9].

The intersection, Q, of S and S_r is a circle whose center lies along the line through the origin, $\mathcal{O}$, and the center of S. As $r \to 0^+$, the center of Q approaches the origin and the radius of Q approaches 0. The curve R is the projection through P of Q onto the $z = 0$ plane. In general, this curve also converges to the single point at the origin.

There is one exception: when the vector from $\mathcal{O}$ to the center of S is perpendicular to $\mathcal{O}P$, i.e., $b = 0$. In this case it is once again possible to obtain explicit formula for the curves Q and R.

Although the parametric forms will not be suitable when it comes time to evaluate the limit as $r \to 0^+$, these representations can be useful in identifying general properties of R.

Let $\theta = \arctan \frac{x}{z} \in (-\pi, \pi]$ denote the angle, measured parallel to the $y = 0$ plane, that a point (x, y, z) makes with the positive z-axis (as described in Section 4). It does not take much effort to see that Q is the circle in the $y = \frac{r^2}{2}$ plane with center $(0, \frac{r^2}{2}, 0)$ and radius $\frac{r}{2}\sqrt{4a^2 - r^2}$. A parametric representation of Q is

$$Q: \quad x = \tfrac{r}{2}\sqrt{4a^2 - r^2}\sin\theta, \quad y = \tfrac{r^2}{2}, \quad z = \tfrac{r}{2}\sqrt{4a^2 - r^2}\cos\theta, \quad \text{for } -\pi < \theta \le \pi$$

Working with this parametric representation we find a parametric representation for R:

$$R: \quad x = \frac{r\sqrt{4a^2-r^2}\sin\theta}{2a-\sqrt{4a^2-r^2}\cos\theta}, \quad y = \frac{r^2}{2a-\sqrt{4a^2-r^2}\cos\theta}, \quad z = 0, \quad \text{for } -\pi < \theta \le \pi$$

The visual evidence suggests that R is also a circle. With this ansatz it is straightforward to show that R is the circle in the $z = 0$ plane with center $(0, 2a, 0)$ and radius $\sqrt{4a^2 - r^2}$.

It is now clear that when $r \to 0^+$, R converges to the circle in the $z = 0$ plane with center $(0, 2a, 0)$ and radius $2|a|$: $x^2 + (y - 2a)^2 = 4a^2$. ■

We are now ready to state and prove the general result for non-spherical fixed surfaces.

Theorem 5 *Let S be a surface that includes the origin and is twice continuously differentiable at the origin. Define S_r, P, Q, and R as in the Generalized Shrinking Sphere Problem. Let the restriction of S to the $x = 0$ plane be denoted by $S|_{x=0}$, let the osculating circle to $S|_{x=0}$ at the origin have radius ρ and center $(0, a, b)$. Then, with $a^2 + b^2 = \rho^2$,*

$$\lim_{r\to 0^+} R = \begin{cases} x^2 + (y - 2\rho)^2 = 4\rho^2 & \text{if } b = 0 \\ (0, 0) & \text{otherwise.} \end{cases}$$

Proof. The proof is an adaptation to three dimensions of the proof of Theorem 2. We focus only on the case when $b = 0$ and $a = \rho \neq 0$. Let $\mathcal{S}_\epsilon$ denote the sphere centered at $(0, \rho + \frac{\epsilon}{2}, 0)$ with radius $\rho + \frac{\epsilon}{2}$. Note that $S|_{x=0}$ is a great circle on $\mathcal{S}_0$. Define Q_ϵ to be the intersection of S_r and $\mathcal{S}_\epsilon$ and define R_ϵ to be the projection from $P = (0, 0, r)$ of Q_ϵ onto the $z = 0$ plane.

Let $\epsilon > 0$ be given. For $r > 0$ but sufficiently close to 0, R is contained within the annulus bounded by the curves R_0 and R_ϵ in the $z = 0$ plane. In the limit as r decreases to 0, R_0 converges to the circle with center $(0, 2\rho, 0)$ and radius 2ρ and R_ϵ converges to the circle with center $(0, 2\rho + \epsilon, 0)$ and radius $2\rho + \epsilon$. And, $\lim_{r\to 0^+} R$ must be in the annulus between these two circles. Since this is true for all $\epsilon > 0$, it must also be true in the limit. Thus, $\lim_{r\to 0^+} R$ is the circle centered at $(0, 2\rho, 0)$ with radius 2ρ. ■

Surface	S	κ	$\lim_{r\to 0^+} R$	Reference
ellipsoid	$\frac{x^2}{a^2} + \left(\frac{y-b}{b}\right)^2 + \frac{z^2}{c^2} = 1$	$\frac{b}{c^2}$	$x^2 + \left(y - \frac{2c^2}{b}\right) = \left(\frac{2c^2}{b}\right)^2$	[10]
cone	$x^2 + (y-a)^2 = (z-b)^2$	0	$(0,0,0)$	[11]
paraboloid	$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{y}{b}$	$\frac{b}{2c^2}$	$x^2 + \left(y - \frac{c^2}{b}\right)^2 = \left(\frac{c^2}{b}\right)^2$	[12]

Table 3: Application of Theorem 5 to non-spherical fixed surfaces. The curvature, κ is the curvature of $S|_{x=0}$.

5.1 Other Geometries

Theorem 5 can be verified for other curves besides a circle. Table 3 shows the results for three general classes of surfaces. The ellipsoid is a natural generalization from the sphere. The cones that are studied have been selected because their restriction to the $x = 0$ plane have zero curvature. The paraboloid extends the special case of the parabola to the three dimensional problem. These cases are complicated by the fact that it is not practical to obtain explicit non-parametric representations for the curves Q and R, yet the limits do converge — to a circle or to the point at the origin.

5.2 Coordinate-Free Version

Theorem 6 *Let $\mathcal{O}$ be a point on a surface S in $\mathbf{R}^3$ with a well-defined normal vector, $\mathbf{N}$, at $\mathcal{O}$. Let C be a curve on S such that, at $\mathcal{O}$, the unit tangent vector to C on S is $\mathbf{T}$ and the principal normal vector for the curve C coincides with the normal vector to S at $\mathcal{O}$, that is, $\mathbf{N} = \left|\frac{d\mathbf{T}}{ds}\right|$ (where s is arclength). For any $r > 0$, define:*

- *S_r to be the sphere with radius r centered at $\mathcal{O}$,*
- *$P = \mathcal{O} + r\mathbf{T}$, the point at the top of S_r,*
- *Q to be the intersection (curve) of S and S_r, and*
- *R to be the curve that is the projection from P through Q onto the plane containing $\mathcal{O}$ that is normal to $\mathbf{T}$.*

Then, as $r \to 0^+$, R converges to the circle with radius $\frac{2}{\kappa}$, centered at $\mathcal{O} + \frac{2}{\kappa}\mathbf{N}$, and lies in the plane with normal vector $\mathbf{T}$.

6 Discussion of Numerical Limits and Visualization

The geometric descriptions of the Shrinking Circle and Shrinking Sphere Problems are easily visualized – in our mind's eye. Creating an animation that accurately captures the limiting process is another matter. In this section the derivations leading to Theorem 2 and Theorem 5 are revisited with the additional objective of producing visualizations consistent with the general theory.

One of the authors began his investigation of the Original Shrinking Circle Problem using ClassPad Manager. The results obtained from this analysis supported a conjecture that the limit would be either infinite or zero [Yang, 2006]. Kreczner reported that *Mathematica* incorrectly evaluates the limit when the fixed curve is a parabola $y^2 = 2ax$ [Kreczner, 1995]. Both of these problem could have been avoided if it had been noticed that the limit in the Shrinking Circle Problem has indeterminate form $\frac{0}{0}$ (Section 2).

The crux of the analysis for the two-dimensional problem is Lemma 1, where C is a circle centered at (a, b) with radius $a^2 + b^2$. Because the limiting value of R exhibits a discontinuity when $b = 0$, any loss of precision in representing the center of C could interfere with numeric and graphic attempts to evaluate this limit.

In three dimensions, the parameterization of Q and R introduced in Section4 is very appealing. But, it is not useful for completing the limits involved in this problem. The special behavior with $\theta = 0$ leads to piecewise-defined (discontinuous) results. These limitations are real. They appear in attempts to create accurate visual representations of the curve R using a finite sampling of points from Q.

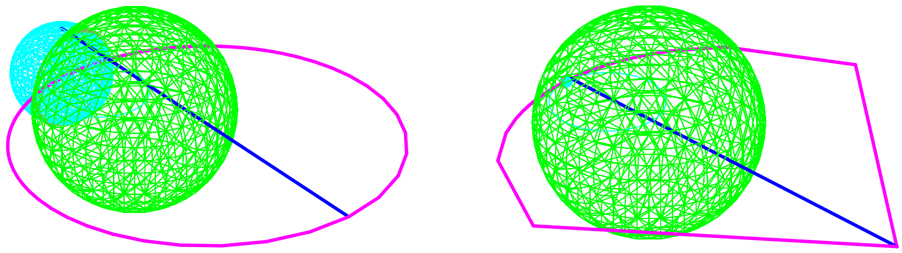
The *Mathematica* experience reported in [Kreczner, 1995] shows that a brute force symbolic attack to the problem is not guaranteed to be successful. The powerful symbolic tools available with a computer algebra system such as Maple or *Mathematica* have to be used with care and intelligence.

The only effective way to work with the curve R (and the curve Q) is to be able to identify the curve geometrically, as was done in the proof of Theorem 4. Dynamic geometry software such as Cabri3D [Cabri3D], ClassPad Manager [ClassPad], Geometer's Sketchpad [GSketchpad], and Geometry Expressions [GExpressions] avoid the analytical indeterminate forms by working directly with the geometric objects — circles, lines, intersections, projections, ...

A full collection of symbolic, numeric, and graphic resources for investigating the Shrinking Circle Problem can be found online at `http://www.math.sc.edu/~meade/eJMT-Shrink/` or `http://www.radford.edu/~scorwin/eJMT/Content/Papers/v1n1p4`.

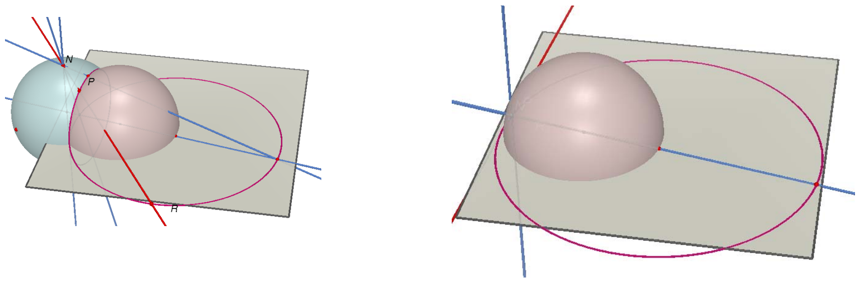
The two-dimensional result (Lemma 1) has direct ramifications for the three-dimensional Shrinking Sphere Problem (Lemma 4). Here, the critical issue is pointwise versus uniform convergence. Recall that Q is the circle formed by the intersection of S, the fixed sphere $x^2 + (y - a)^2 + z^2 = a^2$, and S_r, the sphere $x^2 + y^2 + z^2 = r^2$. The curve R is the projection from P, the top of S_r, onto the $z = 0$ plane. In the proof of Lemma 4 it was shown that R is the circle $x^2 + (y - 2a)^2 = 4a^2 - r^2$.

Difficulties arise when the intersection curve Q is represented by a discrete sampling of points. The point at the apex of Q projects onto the point on R diametrically opposite the origin: $(0, 2a + \sqrt{4a^2 - r^2}, 0)$. As $r \to 0^+$, this point approaches $(4a, 0, 0)$. All other points on R converge to the origin. That is, while the (uniform) limit of the projected circles is a circle, the pointwise limit of the projected circles is one of two points: the origin and its diametric opposite, viz., $(0, 0, 0)$ and $(0, 4a, 0)$.



(a) (b)

Figure 5: Shrinking Sphere Problem with (a) S_1 and (b) $S_{\frac{1}{10}}$. Even though the plots of Q are formed with 201 points, the lack of uniform convergence is particularly evident in (b). (Plots produced with [Maple], [13].)



(a) (b)

Figure 6: Shrinking Sphere Problem. These plots are smoother because they are constructed from the geometry, not from a sampling of points. (Plots produced with [Cabri3D], [15].)

When S is not a sphere, Q and R might not be circles, but R converges to a circle (or a point). If, by coincidence, it is possible to provide a geometric description of Q and R then there is a chance to construct a decent animation showing the convergence of R to a circle. If Q and R cannot be identified, then a discrete sampling of data points can be used effectively for moderately large values of r, but not as r nears zero.

Figures 5 and 6 show two snapshots from an animation created in Maple and Cabri3D, respectively. In each plot in Figure 5, the curve Q is represented by a uniform sample of 201 points. The rapid convergence of all points (except one) to the origin is very apparent. Using more points to represent Q is a possibility, but is not without additional costs in time and computing. The plots shown in Figure 6 are better because they are created from the geometry of the problem and not merely a collection of points.

Dynamic geometry software would appear to be the most appropriate choice of software to use to create effective visualizations of the Shrinking Sphere Problem. Unfortunately, dynamic geometry software is only now starting to move into the third dimension and has very little to offer in the way of working with general surfaces, or even ellipsoids, cones, and paraboloids.

7 Conclusion

The Generalized Shrinking Circle and Generalized Shrinking Sphere Problems have been analyzed, with general results stated in terms of properties of an appropriate osculating circle. The proofs involve a combination of symbolic manipulation and analytic geometry. The results have been confirmed on other surfaces, and coordinate-free versions are provided.

These results have a strong geometric flavor that beg to be visualized and animated. The discussion identifies some of the mathematical obstacles to obtaining reliable computer-generated numeric, graphic, or symbolic results. In particular, it was shown how indeterminate forms can compromise the reliability of numerical computations and how the absence of uniform convergence with respect to a parameterization of a curve can lead to misleading symbolic and graphical results.

While these obstacles have been overcome for the problems considered in this paper, they do point to the need for the development of better software tools for dynamic geometry in both two and three dimensions.

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Software Packages

[Cabri3D] Cabri 3D, a product of CABRILOG, `http://www.cabri.com/v2/pages/en/index.php`.

[ClassPad] ClassPad Manager, a product of CASIO Computer Ltd. `http://www.classpad.org/`.

[GExpressions] Geometry Expressions 1.0.24, a product of Saltire Software, 2006 `http://www.geometryexpressions.com/`.

[GSketchpad] Geometer's Sketchpad v. 4, a product of Key Curriculum Press, 2006 http://www.dynamicgeometry.com/.

[Maple] Maple 10.0.6, a product of Maplesoft, 2005 http://www.maplesoft.com/.

[Mathematica] Mathematica, a product of Wolfram Research, Inc., http://www.wolfram.com/.

Supplemental Electronic Materials

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