Alternative Solutions and Comments to the Problem Corner – October 2020 issue

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Summary: We provide alternative solutions by using *GeoGebra Discovery* for both Problems 1 and 2 (Problem Corner, Oct. 2020). We also correct one of the statements of Problem 2. An interactive version of our solutions is available at https://matek.hu/zoltan/pc20oct.php.

Let *Q* be a convex quadrilateral with vertices A, B, C, D.

We call edges of *Q* the four sides and the two diagonals, AB, BC, CD, DA, AC, BD.

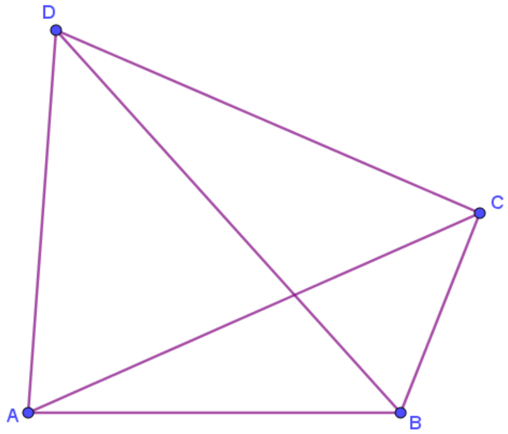

Figure 1. The quadrilateral *Q*

Problem 1

Let M1, M2, M3, M4, M5, M6 be the midpoints of the edges AB, BC, CD, DA, AC, BD.

Prove that the segments M1M3, M2M4, M5M6 are concurrent in a point G that bisects them all.

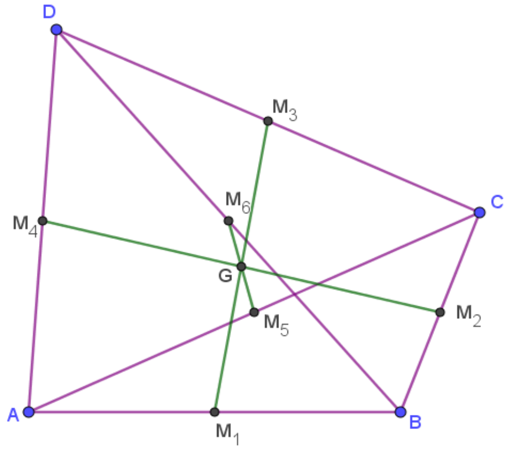

Figure 2. *Q* and the midpoint segments

SOLUTION

Hi, I am GeoGebra Discovery[1]*, a new fork of GeoGebra that is able to work as an automated geometer*, see [Kovács-Recio, 2020]. This is the way GeoGebra Discovery can address Problem 1.

Figure 1 below shows the image of the quadrilateral constructed with GeoGebra tools: M1 is the midpoint of side AB, M2 is the midpoint of BC, M3 the midpoint of side CD and M4 the midpoint of side DA. Moreover, M5, midpoint of the AC, and M6, midpoint of the diagonal DB are created. Next, h is the segment M1M3, and i is the segment M2M4. The crucial point G is built as the intersection of h and i.

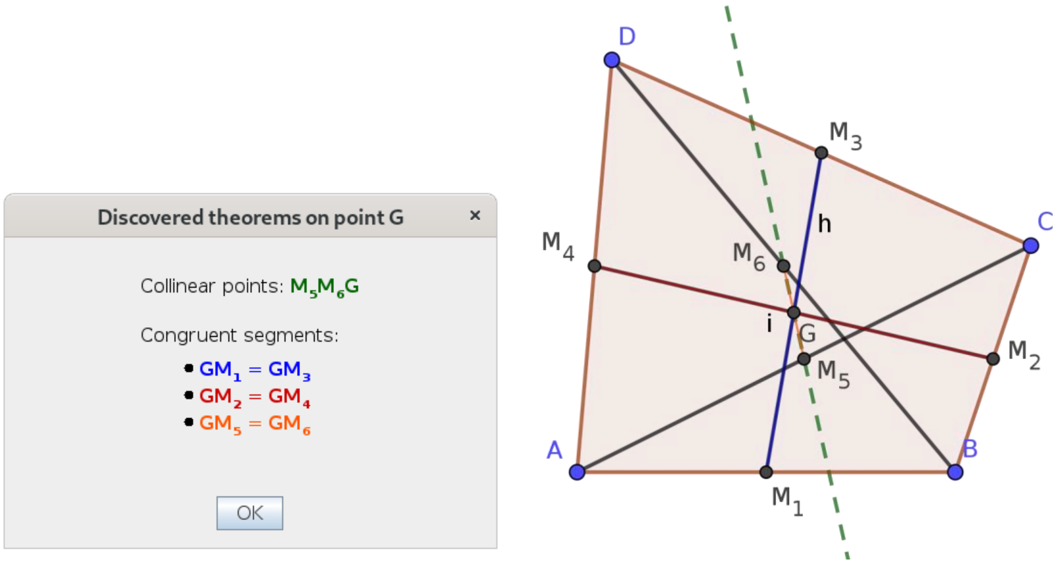

Figure 3

1 https://github.com/kovzol/geogebra-discovery#readme
https://github.com/kovzol/geogebra/releases

Now we simply ask GeoGebra to Discover(G), that is, to discover properties holding (from a mathematically rigorous point of view, that is, not numerical, not just for this figure, but considering all possible positions of the vertices of the quadrilateral, etc.) and involving point G.

The practically immediate answer (using GeoGebra Discovery over an old MacBookPro of 2015) is what appears in Figure 3, where the Discover command automatically displays some lines and colors to make easier to understand what it has obtained. In particular, GeoGebra finds, without requiring the user to pose the statement, that M5M6G are collinear (or, with the notation of the Problem 1, that "the segments M1M3, M2M4, M5M6 are concurrent")!

Moreover, GeoGebra Discovery finds as well that GM1=GM3, GM2=GM4, GM5=GM6, implying that G is the midpoint of the corresponding segments, that is, solving the last statement of Problem 1 "M1M3, M2M4, M5M6 are concurrent… in a point G that bisects them all."

Problem 2

Let A', B', C' and D' be the centroids of the triangles BCD, ACD, ABD and ABC respectively.

Prove that

– the segments AA', BB', CC' and DD' are concurrent in G;

– G divides each segment in two parts, the one containing the vertex twice the other one.

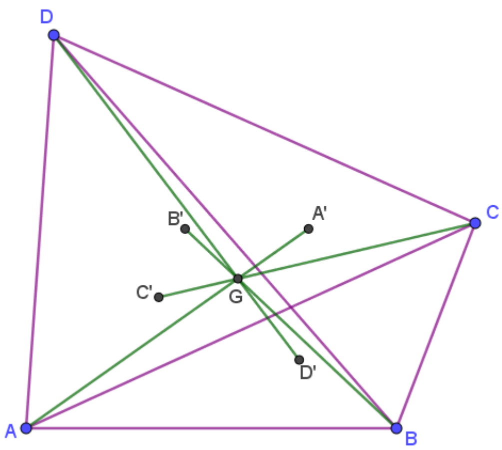

Figure 4. *Q* and centroid segments

SOLUTION

Figure 5 deals with the second problem. Problem 2 asks, first, to prove the concurrency of four segments described from the vertices to the centroids of some triangles. In the GeoGebra construction we have built the centroid A' of BCD, B' of ACD, C' of ABD and D' of ABC. For example, A' has been created by the intersection of DM2 and BM3, denoted by j and k, respectively. Moreover, GeoGebra has labelled as l the segment AA', m=segment BB', n=segment CC' (and another segment could be drawn for DD'). The first question of Problem 2 asks to prove that, for example, the segments l, m and n are concurrent (the concurrency of some other triple of segments, including the fourth one, could be done likewise).

This has been proved automatically by GeoGebra Discovery in Figure 5 by using the command Relation({l,m,n}) to prove the concurrency of the first set of three segments, yielding a positive answer, except when the quadrilateral degenerates (e.g. the four vertices are aligned, two vertices coincide, etc.).

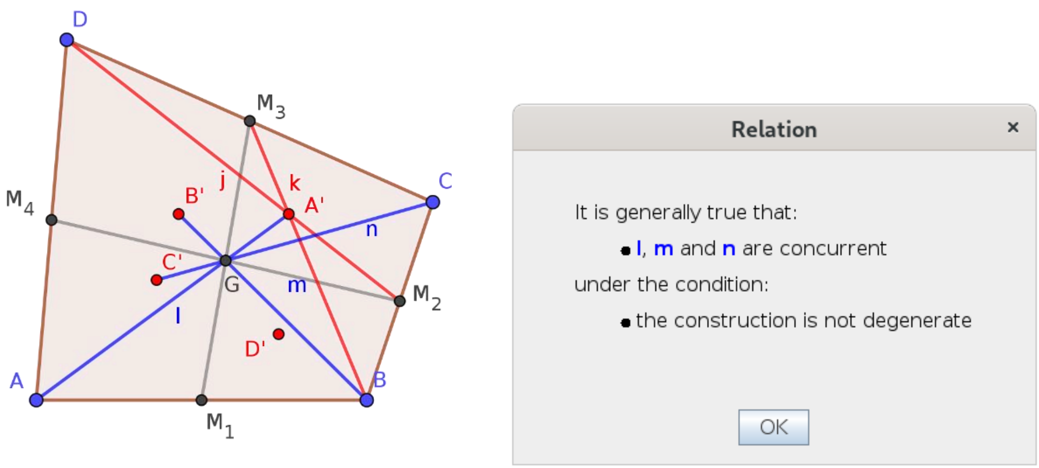

Figure 5

We also need to prove that the point of concurrency is G. To achieve it we select the Relation tool and click on point G and segment l to get a positive answer (see Figure 6). We note that the list of non-degeneracy conditions includes that r=AC and q=BD needs to intersect (something obvious for convex quadrilaterals) and two other simple assumptions.

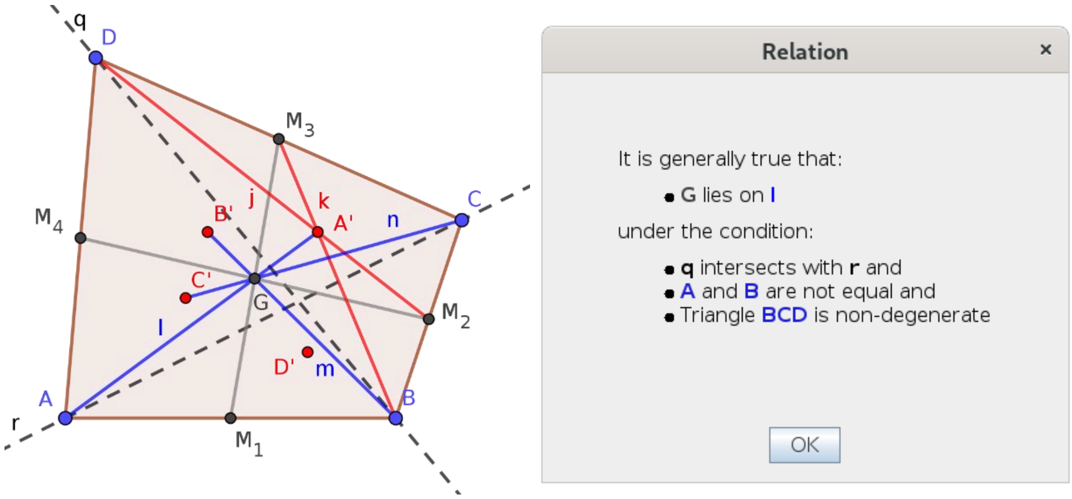

Figure 6

Finally, the second problem asks to prove that the four concurrent segments, meeting at point G, have the property that G divides each of the above segments in two parts, one of *three* times the size of the other. Here, as shown in Figure 7, GeoGebra is asked, through the Relation command, to prove that $3 \cdot s = t$, where s is segment A'G and t is segment AG. The same comparison could be done for the other segments as well.

The answer is that the result is "true on parts, false on parts" because the concept of "segment length" is internally translated by GeoGebra as the square root of some polynomial, so the truth or failure of the statement depends on the chosen sign of the square root, according to the most suitable interpretation in the given statement. Obviously, in the most common case, as in this problem, for positive lengths, the statement is true! See [Kovács, Recio and Vélez, 2019] for a complete description of this involved issue.

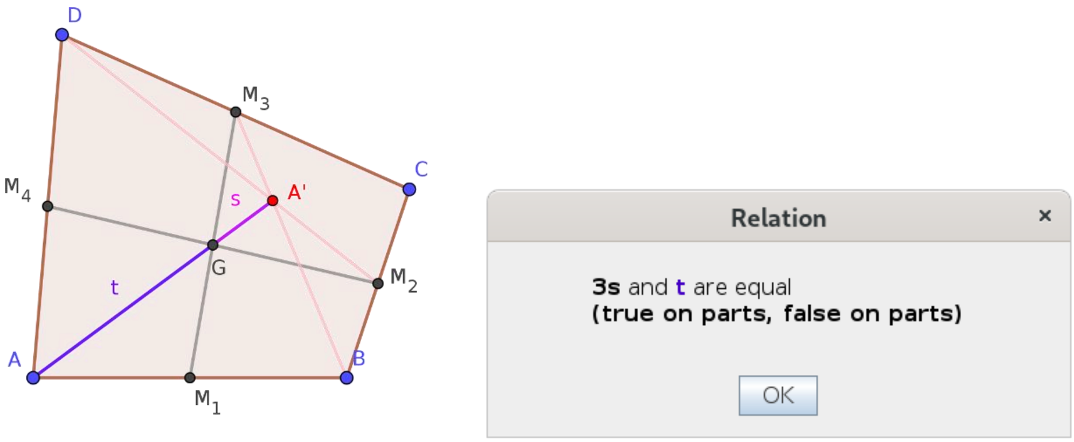

Figure 7

REFERENCES

Kovács, Z. and Recio, T.: "GeoGebra reasoning tools for humans and for automatons." Electronic Proceedings of the 25th Asian Technology Conference in Mathematics, December 14-16, 2020. ISSN 1940-4204 (online version). http://atcm.mathandtech.org/EP2020/invited/21786.pdf

Kovács, Z., Recio, T. and Vélez, M.P.: "Detecting truth, just on parts." Revista Matemática Complutense, Volume 32, Issue 2, May 2019. pp. 451-474. DOI: 10.1007/s13163-018-0286-1