Problem Corner:  
Interesting Numerical Differentiation Tidbits

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Recall from the last Problem Corner that finite differences can be used to approximate values of the derivative of a differentiable function \( f(x) \). Using the forward difference approximation

\[
 f'(x) \approx \frac{f(x+h) - f(x)}{h},
\]

we can approximate \( f'(x) \) using suitably chosen values of \( h \).

We saw that the magnitude of the actual error in our approximations has the form

\[
 E(h) = Ah + \frac{B}{h}
\]

where \( A \) and \( B \) are positive constants. We further saw that \( A \) and \( B \) can be estimated using linear regression.

When two point centered differences are used to approximate \( f'(x) \) or three point centered differences are used to approximate \( f''(x) \), we obtain similar expressions

\[
 E(h) = Ah^2 + \frac{B}{h}
\]

since the truncation errors for these methods are second order accurate. For either of these two methods, \( A \) and \( B \) can be estimated using linear regression. Refer to the attached Maple worksheets fdiv2.mws and fdiv3.mws for details. Proving that we are led to a nonsingular system of linear equations is more difficult for these methods.

More generally, suppose we wish to fit a set of data with a fit of the form

\[
 E(x) = Af(x) + Bg(x).
\]

This form obviously encompasses each of the three finite difference error expressions. Suppose the data in question is \( (x_i, y_i), i = 1, \ldots, n \) and we wish to approximate \( A \) and \( B \) by minimizing the sum of squares

\[
 r(A, B) = \sum_{i=1}^{n} (Af(x_i) + Bg(x_i) - y_i)^2.
\]

As before calculate the partials of \( r(x) \) with respect to \( A \) and \( B \) and equate them to 0 to obtain a system of linear equations. Find a simple condition that characterizes the sets of data for which this system of equations is nonsingular and show that it holds for each of the three finite difference methods.