Problem 1
Consider parallelogram ABCD, \(|AB| \neq |BC|\). Let E be the intersection of the perpendicular to the diagonal AC dropped from the point D with the line BC and let F be the foot of the perpendicular from the point B to the line DE. Assuming that the lines CF and AE perpendicular, determine the angle ACB.

SOLUTION
The point F is the orthocenter of the triangle AEC, hence AF is perpendicular to BC. As the segments AD and BC are equal and parallel, the diagonal AC is a bisector of the segment DF and the triangle DAF is isosceles. From that follows

\[ \angle ACB = \angle DAC = \angle CAF \]

Hence \( \angle ACB = 45^\circ \).
Problem 2
Consider a triangle $ABC$ and its circumscribed circle $k$. On the circle choose an arbitrary point $P$ and inside the triangle select an arbitrary point $G$. Consider circles $GAB$, $GBC$, $GCA$. Denoting $P_{AB}$, $P_{BC}$, $P_{CA}$ the inverse images of the $P$ with respect to the circles, prove or answer following statements:

a) Points $P_{AB}$, $P_{BC}$, $P_{CA}$ and $G$ lie on a circle $C$.
b) As $P$ moves along the circle $k$, the centre of the circle $C$ moves along a line.
c) Determine in the triangle a point $G = G_L$ in such a way that the points $P_{AB}$, $P_{BC}$, $P_{CA}$ and $G_L$ are always collinear (we consider a line as a special case of a circle).

Hint: Apply the Simson-Wallace theorem.

Solution

a) Let us start by the Simson-Wallace Theorem [1]:

Let $ABC$ be a triangle and $P$ a point in a plane. If $P$ lies on the triangle’s circumcircle, then its images in reflection with respect to the sides of the triangle $ABC$ and the orthocentre of the triangle are collinear – they lie on the so-called Simson line.

Let consider inverse image of the geometrical objects from the task with respect to an arbitrary circle with the centre $G$. Because the relation “point – inverse circle – image” is preserved by inverse mapping, we can reformulate the problem exactly as a statement of S.-W. theorem: The three circles are mapped to lines, the points $P, P_{AB}, P_{BC}, P_{CA}$ are mapped to points $P', P'_{AB}, P'_{BC}, P'_{CA}$ and the points $P'_{AB}, P'_{BC}, P'_{CA}$ are reflections of the point $P'$ with respect to these lines. Because the point $P$ lies on the circumscribed circle of the inverse image of the triangle $ABC$, the points $P'_{AB}, P'_{BC}, P'_{CA}$ lie on a Simson line. Further, the image of the point $G$ lies in infinity and hence lies on the line too. As the inverse images of the four points lie on a line, the original points $P_{AB}, P_{BC}, P_{CA}, G$ must lie on a circle $C$ or on a line, which is a special case of circle.

b) The most straightforward way is to prove that the circle $C$ passes through two fixed points. The first point is obviously $G$. From the S.-W. theorem, we know that the Simson line passes through the orthocentre $H$, hence its image – the circle $C$ – passes through the inverse image of $H$. It is the second point.

c) We will again use the inverse images of the objects from the task: The Simson line passes through the orthocentre $H$. In order to preserve this line
under mapping, it is necessary $G_L = H$, hence $G_L$ must be the orthocentre of the image of the original triangle ABC. Consider points A, B and its inverse images A’, B’. Let the centrum of an inverse circle be a point G. As is obvious from fig 2, the angles are swapped by the mapping (justification is left to the reader).

From that follows that the point $G_L$ is the orthocentre of the inverse imagine of the triangle ABC if and only if it is the incentre or excentre of the triangle ABC. As the point $G_L$ is inside the triangle, it is its incentre.

**Problem 3**

On a circle $k$ are arbitrarily selected points $A, B, C, D$. Denote the orthocenter of the triangle $ABC$ as $H_D$ and analogically introduce the orthocenters $H_A, H_B, H_C$. Prove that the orthocenters lie on a circle with its radius equal to the radius of the circle $k$.

**SOLUTION**

We will solve the problem by means of vectors. Let the center of the $k$ is $O$ and let it be the origin of the coordinate system. It is easy to show that

$$H_D = \vec{A} + \vec{B} + \vec{C} = (\vec{A} + \vec{B} + \vec{C} + \vec{D}) - \vec{D}$$

and that the point $H_D$ lies on the circle with the center $S = (\vec{A} + \vec{B} + \vec{C} + \vec{D})$ and radius $R = |\vec{D}| = |\vec{A}| = |\vec{B}| = |\vec{C}|$.

For the other three points we arrive at the same conclusion.