PROBLEM CORNER

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Let \( Q \) be a convex quadrilateral with vertices A, B, C, D.

We call edges of \( Q \) the four sides and the two diagonals, \( AB, BC, CD, DA, AC, BD \).

Problem 1

Let \( M_1, M_2, M_3, M_4, M_5, M_6 \) be the midpoints of the edges \( AB, BC, CD, DA, AC, BD \).

Prove that the segments \( M_1M_3, M_2M_4, M_5M_6 \) are concurrent in a point G that bisects them all.

SOLUTION

In the triangle ABC, the segment \( M_1M_2 \) joins the midpoints of the edges \( AB \) and \( BC \), then \( M_1M_2 \) is parallel to \( AC \) and \( M_1M_2 = \frac{1}{2} AC \). Analogously, the segment \( M_3M_4 \) is parallel to
AC and $M_3M_4 = \frac{1}{2} AC$. Therefore, $M_1M_2M_3M_4$ is a parallelogram. The common point $G$ of its diagonals bisects both of them, $M_1M_3$ and $M_2M_4$.

![Figure 3](image)

Figure 3. $Q$ and the parallelogram $M_1M_2M_3M_4$

Let us consider different cases on $Q$.

**Case 1.** $Q$ does not have any pairs of opposite parallel sides.

In the triangle $ABC$, the segment $M_1M_5$ joins the midpoints of the edges $AB$ and $AC$, then $M_1M_5$ is parallel to $BC$ and $M_1M_5 = \frac{1}{2} BC$. Analogously, the segment $M_3M_6$ is parallel to $BC$ and $M_3M_6 = \frac{1}{2} BC$. Therefore, $M_1M_5M_3M_6$ is a parallelogram. Its diagonals bisect each other and since $G$ is the midpoint of $M_1M_3$ then $G$ is also the midpoint of $M_5M_6$.

Therefore, the segments $M_1M_3$, $M_2M_4$, $M_5M_6$ are concurrent in a point $G$ that bisects them all.

Observe that also the quadrilateral $M_4M_5M_2M_6$ is a parallelogram.

![Figure 4](image)

Figure 4. $Q$ in case 1

**Case 2.** $Q$ has exactly one pair of opposite parallel sides.
Assume that AB is parallel to CD.

\( M_4M_5M_6 \) does not exist anymore (because the segments \( M_2M_6 \) and \( M_4M_5 \) are parallel to BC and the segments \( M_4M_6 \) and \( M_5M_2 \) are parallel to AB. Since AB is parallel to CD they are all parallel to each other, therefore the points \( M_2, M_5, M_4 \) and \( M_6 \) are collinear and \( M_5M_6 \) is contained in \( M_2M_4 \), but the parallelograms \( M_1M_2M_3M_4 \) and \( M_1M_5M_3M_6 \) still hold and, since they share the diagonal \( M_1M_3 \) then they all meet in a point \( G \) that bisects \( M_1M_3, M_2M_4, M_5M_6 \).

![Figure 5. \( Q \) in case 2](image)

**Case 3.** \( Q \) is a parallelogram.

If \( Q \) is a parallelogram then the parallelograms \( M_4M_5M_2M_6 \) and \( M_1M_5M_3M_6 \) do not exist anymore because \( M_5 \) and \( M_6 \) coincide with \( G \) (being \( G \) midpoint of the diagonals AC and BD).

Then the problem is solved also in this case.

![Figure 6. \( Q \) in case 3](image)

**Problem 2**

Let \( A', B', C' \) and \( D' \) be the centroids of the triangles BCD, ACD, ABD and ABC respectively.

Prove that

- the segments \( AA', BB', CC' \) and \( DD' \) are concurrent in \( G \);
- \( G \) divides each segment in two parts, the one containing the vertex twice the other one.
SOLUTION
Let $M_2$ be the midpoint of $BC$. The segment $DM_2$ is a median of the triangle $BCD$, therefore it contains the centroid $A'$. Let $N$ be the midpoint of $DA'$ and $M_4$ the midpoint of $AD$. The segment $NM_4$ joins the midpoints of the edges $DA'$ and $DA$ of the triangle $DAA'$, then $NM_4$ is parallel to $AA'$ and $AA' = 2 \ NM_4$.

Let $G$ be the common point of $AA'$ and $M_2M_4$. Let us prove that $G$ is the midpoint of $M_2M_4$. In fact, the segment $GA'$ is parallel to $NM_4$ and passes through the midpoint $A'$ of the edge $NM_2$ of the triangle $NM_2M_4$, then $G$ is the midpoint of $M_2M_4$. Moreover it is $NM_4 = 2GA'$, and then $AA' = 2NM_4 = 4GA'$ and $AG = 3GA'$.

Therefore $G$ lies on the segment $AA'$ and it is such that $AG = 3GA'$; the same holds for the segments $BB'$, $CC'$, $DD'$ and it is $BG = 3GB'$, $CG = 3GC'$, $DG = 3GD'$ (in the proof you should consider the segments $M_1M_3$, $M_2M_4$ and $M_1M_3$ respectively). Note that the point $G$ bisects the two segments $M_1M_3$, $M_2M_4$ and therefore is the same point $G$ as in Problem 1. This point is known as centroid of a quadrilateral.