

Exploration of envelopes of parameterized families of surfaces in a technology-rich environment

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Abstract

The study of envelopes requires a strong dialog between algebraic computations and graphic representations. Both are available using a Computer Algebra System, and sometimes accompanied by a Dynamic Geometry System. The two kinds of software have different affordances, whence different characteristics of the animations. Visualization raises specific problems. We illustrate this with examples of 1-parameter and 2-parameter families of surfaces in 3D space. In certain cases, the envelopes have non-isolated singularities; we study them switching between parametric and implicit presentations. Finally, networking between kinds of software and between visualization and symbolic computations is discussed.

1 Introduction

Envelopes of parametric families of plane curves, and of parametric families of surfaces have been studied for a long time, but disappeared from the basic syllabus decades ago; see Thom's remarks about that in 1962 [41]. Actually, the topic has never been abandoned, because of its centrality in numerous fields, such as robotics and safety zones in industrial plants or entertainment parks [11], pollution and depollution of soils [44]. The introduction of mathematical software, both for research and for teaching, and other technologies, enabled a revival of the topic for a more general audience than the specialists, and at an early stage of education. This is the goal in [19], focusing on families of plane curves. Nevertheless, they continued to be studied by researchers. Easy examples can be presented to students, even very young, using threads and nails. See, for example Figure 1; it shows an astroid as both a creation with threads (what is called string art) and a model with GeoGebra. Such concrete realizations may be easy to construct, but with more advanced examples, the problem of visualization is acute. Note that the concrete realization shows actually a surface in 3D because of the thickness of the threads. In order to get rid of the problem, we have to look at the artistic creation from above. Note also that the intuitive meaning of the word *envelope* may be misleading. An envelope, in the sense of what we will see later, is not a wrapping. In Figure 1, the astroid is realized as the

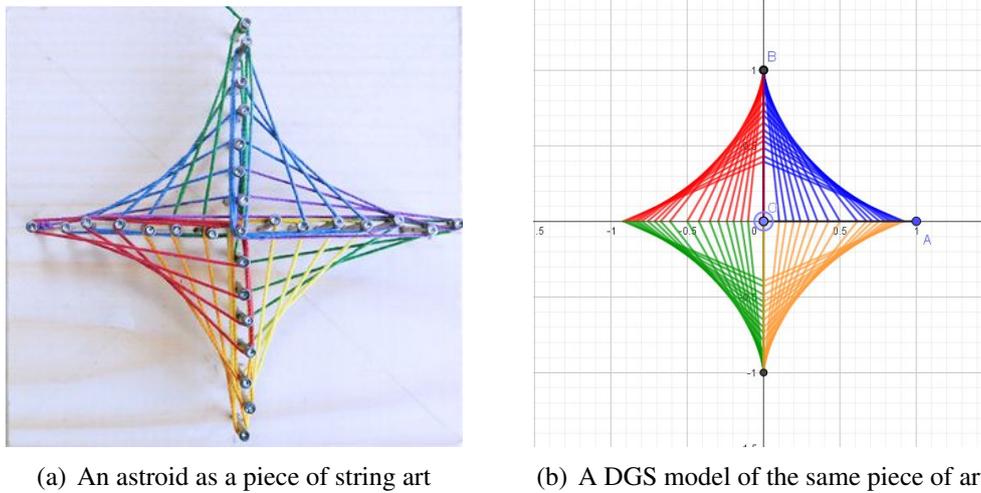


Figure 1: String art and a DGS model

envelope of segments of length 1 whose endpoints are on the axes. Here the envelope wraps the elements of the family, but this not always the case [11, 12, 18, 20]; see the examples developed in Section 4.

Figure 1(b) has been obtained with a [GeoGebra online applet](#). The construction is purely geometric and uses dragging with **Trace On**. This construction can be generalized to 3D in various ways; one of them is described in an [online applet](#) and a snapshot of a partial construction is shown in Figure 2 (pay attention to the astroid in the xy -plane in the 3D plot: the 2D window and the 3D window are totally synchronized). A detailed study of this surface is beyond the scope of this paper, as it requires the definition of an envelope of a family of line segments in space.

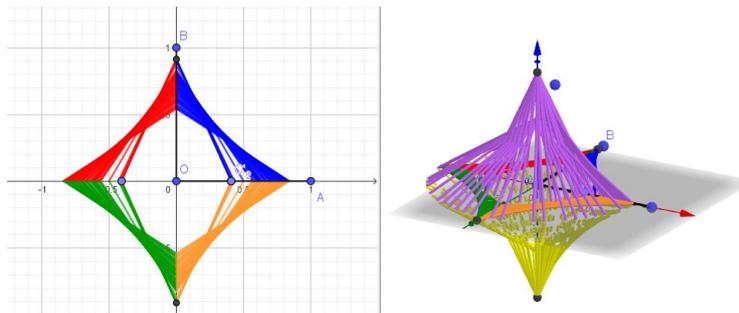


Figure 2: A partial construction of a 3D generalization of an astroid as an envelope of segments

The surface "looks like" a Beltrami surface (see Figure 3 and worksheet [S1]), which provides a model for the hyperbolic plane. It is given by the following parametrization:

$$\begin{cases} x = \frac{\cos v}{\cosh u} \\ y = \frac{\sin v}{\cosh u} \\ z = u - \tanh u \end{cases} \quad (1)$$

where $u \in \mathbb{R}$ and $v \in [0, 2\pi]$. In fact, it is easy to prove that this is not the surface we explore above.

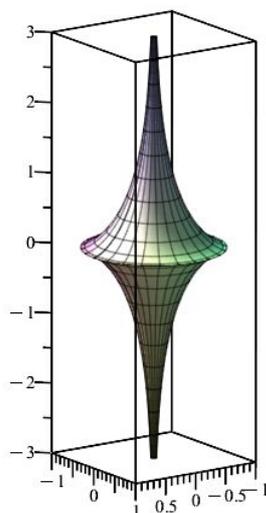


Figure 3: Pseudo-sphere aka. Beltrami surface

Automated methods for the exploration of envelopes have been developed and implemented in software, for example in [GeoGebra-Discovery](#), a companion package of new commands for GeoGebra, for automated exploration, discovery and proofs (see Section 2). They have been developed for envelopes of families of plane curves; from personal discussions, it is reasonable to hope to see in the next future such automated commands on 3D too. In the present paper, we explore envelopes of parameterized families of surfaces in \mathbb{R}^3 , therefore we cannot use existing automated tools for 2D, but we use other features available in CAS (here Maple 2022) and DGS (here GeoGebra). Note that a CAS has strong plotting features, but not with the same dynamics as a DGS, and a DGS may contain a CAS component [28], but generally not as powerful as a CAS. With both kinds of software, we use commands such as **solve** to solve systems of equations (either polynomial or not). This may be transparent to the user, with the drawback of using the CAS as a blackbox. We use the algebraic algorithms for exact computations, implicitization of the parametric presentations, with elimination and Gröbner packages, and of graphical features, for plotting and animating the plots. Working together with a CAS and a DGS as in [18] offers opportunities to build bridges towards visual arts. In certain occurrences, exact implicitization is impossible; for this approximate methods have been explored and developed [40].

New affordances of the software induce always an interest for new tasks [26, 19], including new proofs of classical results. In particular, the possibilities offered by dragging points directly on the screen is a powerful tool for exploration and expressing conjectures; see [3]. Nevertheless, dragging may not yield easily understood results in 3D as in 2D (see subsection 4.2). An analysis of the construction of such new tasks, enhancing a new working scheme based on exploration, is developed in [24]. Exploration leads to conjectures, and eventually to discovery and to proofs. Once again, they have mostly been developed and documented in 2D, but some of them can be adapted for 3D. In this paper we relate to the same old-new topic: envelopes of parameterized families of surfaces. An example is briefly presented at the beginning of Section 2.

On another level, the name of the game is given by the 4 C's of 21st education, namely Collaboration, Communication, Critical Thinking and Creativity [31, 34, 38]. If most of the time, people think about the first two C's as between humans, in fact they exist also between machines and between humans and machines; see [13]. The two last C's are purely inter-humans.

The insertion of Arts in STEM education offers ways to develop creativity; this point is discussed in [16]. The string art displayed above in Figure 1 is such an example. Figure 4 is a photo (from the Wikipedia page) of the Guggenheim Museum in Bilbao, Spain. The general shape and parts of the buildings, namely the cuspidal edges, will evoke some details studied in Section 2.



Figure 4: Guggenheim Museum, Bilbao, Spain

In fact, we may point to students' creativity but also, and first, to educators' creativity, in order to propose modern applications of their teaching. Numerous occurrences of mathematics in situation can be proposed, as briefly shown in [11] for safety zones in industrial plants where robots are at work, or in entertainment parks. Based on [4] and [42], Barabash and Naftaliev analyze in [6] the changes in definitions and in the construction of mathematical meaning induced by technology. Understanding the different definitions of an envelope (Section 2), is a good example.

After [20, 21], the present paper proposes a contribution to the exploration of envelopes of parameterized families of surfaces. Actually it reports on ways for the technology-assisted study of envelopes. In most examples, computations are fully presented, but sometimes they are not, in order not to increase the length of the paper. In a few cases, we have to leave the computations open as the CAS could not give an answer in what we can call a reasonable time. We consider these questions as open and plan to present a solution in a subsequent paper. In Section 4.1, we explore a family of spheres, its envelope and the singular locus of this envelope, comparing this with a similar situation in 2D.

The present paper is based on an invited lecture at the ACTM 2022 conference in Prague (Czech Republic), December 2022.

2 Envelopes of families of plane curves

The study of envelopes of 1-parameter families of plane curves, including working with technology, is well-developed and documented, and important developments have been obtained during the last decade; see [9, 19, 8]. For 1-parameter families of plane curves, exploration with a Dynamic Geometry System (DGS) has proven to be very fruitful and efficient. Tools have been developed for

automated exploration and determination of envelopes; they are part of the package *GeoGebra Discovery*. A guide to the package is given in [32]. The commands **Envelope** and **Locus** are central tools for this automated exploration; see [29, 33] for a general description of the automated reasoning tools (ART) of GeoGebra. The dragging together with the Trace On feature have proven efficient for the exploration of the special case of envelopes in 3D called canal surfaces [14], but in general the center of gravity moves towards the CAS. For a general discussion of automated tools in education, see also [35].

There exist several different and non-equivalent definitions of an envelope in the literature. For 1-parameter families of plane curves, they have been explained by [27, 9] and the relations between the different kinds of definitions have been settled. Plotting technologies help to visualize the differences between the definitions. One of these 4 definitions of an envelope, the only one used by Berger [7], and which has been used in [19], reads as follows:

Definition 1 Let C_k be a family of plane curves given by an equation $F(x, y, k) = 0$, where k is a real parameter. An envelope \mathcal{D} of the family is the curve determined by the solutions of the system of equations

$$\begin{cases} F(x, y, k) = 0 \\ \frac{\partial}{\partial k} F(x, y, k) = 0 \end{cases} \quad (2)$$

This is the definition that we use in this work.

The present example will serve both as an introductory one, and as the basis of what is shown in subsection 4.3. Consider the circle \mathcal{C} centered at the origin with radius 1, and the family of circles centered on \mathcal{C} and tangent to the x -axis. Figure 5 is a snapshot of a [GeoGebra applet](#): one of the circles of the family is enhanced and dotted, and the nephroid is plotted with a thick line. Computations for such an example are given in [19]. The **Envelope** command of GeoGebra provides both a plot and a

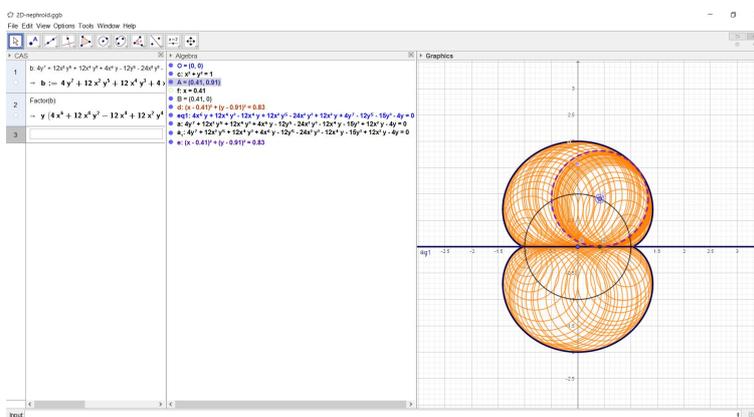


Figure 5: A nephroid as an envelope

polynomial equation, namely:

$$4y^7 + 12x^2y^5 + 12x^4y^3 + 4x^6y - 12y^5 - 24x^2y^3 - 12x^4y - 15y^3 + 12x^2y - 4y = 0 \quad (3)$$

The polynomial in left-hand side is reducible: it is the product of y by a polynomial of degree 6, which gives the equation of a nephroid. The factor y describes the x -axis, which is not surprising as all the

circles are, by definition, tangent to this axis. Nevertheless, only a segment of this axis is actually an envelope of the given family of circles; the x -axis is the Zariski closure of the required segment, and that is what the CAS actually computes. This discussion can be avoided using a trigonometric parametric presentation for the objects involved in the construction.

The factorization of the 7th degree polynomial has been performed with the CAS implemented in GeoGebra. This step was not a must here, as the factor y is easily identified. We mention the possibility to factor using the CAS, as in other cases, factorization may be non-trivial and then the CAS may be helpful. This step is important to explore whether the obtained variety is reducible or not. Recall that as work is performed over \mathbb{R} , it happens that the obtained variety is the union of two disjoint components, but these are impossible to distinguish by algebraic means. For example, it happens with isoptic curves of conics [22, 23].

3 Envelopes of families of surfaces

The translation of Definition 1 to a 1-parameter family of surfaces in 3D-space reads as follows:

Definition 2 *Let \mathcal{S}_a be a family of surfaces in the 3D-space given by an equation of the form $F(x, y, z, a) = 0$, where a is a real parameter. If it exists, an envelope of this family is the surface given by the solutions of the following system of equations:*

$$\begin{cases} F(x, y, z, u) = 0 \\ \frac{\partial F}{\partial u} F(x, y, z, u) = 0 \end{cases} \quad (4)$$

Adapting two other definitions from [9] to 3D yields the following definitions. We consider a 1-parameter family of surfaces S_u , $u \in \mathbb{R}$.

Definition 3 *The envelope \mathcal{E}_1 is the union of the limits of intersections of infinitesimally closed surfaces S_u and $S_{u+\varepsilon}$.*

Definition 4 *The envelope \mathcal{E}_2 is a surface tangent to all the surfaces S_u .*

This is the 3D version of the definition used in [33], for their **Envelope** command. recall that we used it for the nephroid in Section 2.

It can be easily shown that $\mathcal{E}_1 \subset \mathcal{D}$ and $\mathcal{E}_2 \subset \mathcal{D}$. A 2D case where the inclusion is not an equality, i.e. $\mathcal{E}_2 \neq \mathcal{D}$ is explored in [12].

Families of planes have been studied in [19, 20]. In general, 1-parameter families of planes have an envelope, with the following properties:

- The envelope is a ruled surface.
- The envelope has singular points forming a cuspidal edge.
- The tangents to this cuspidal edge are generators of the envelope.

3.1 First example of a family of planes: polynomial coefficients

Consider the family of planes given by the polynomial $F(x, y, u) = x + uy + 2u^2z - 3u^3$, where u is a real parameter. The envelope obtained according to Definition 2 has the following parametric presentation

$$\begin{cases} x = 2u^2t - 6u^3 \\ y = -4ut + 9u^2 \\ z = t \end{cases} \quad (5)$$

and the following implicit equation

$$32xz^3 - 4y^2z^2 + 108xyz - 12y^3 + 243x^2 = 0. \quad (6)$$

Equation (9) has been obtained from Equations (8) by elimination of the parameters. Three views of the envelope are shown in Figure 7 in order to emphasize its remarkable features. In the first one, the cuspidal edge is enhanced. The cuspidal edge, i.e., the set of non-isolated singularities is the curve

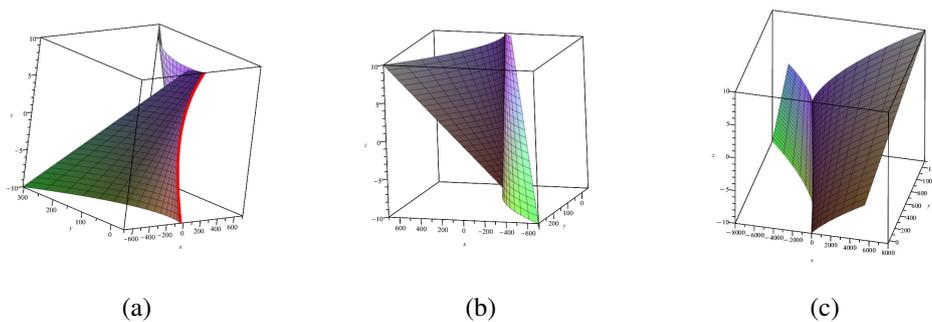


Figure 6: The envelope of the family of planes with equation $x + uy + 2u^2z - 3u^3 = 0$

determined by the following parametric equations:

$$\begin{cases} x = \frac{8t^3}{243} \\ y = -\frac{4t^2}{9} \\ z = t \end{cases} \quad (7)$$

It can be obtained by various ways, either computing separately the partial derivatives of the left hand side of Equation (9) and looking for their common vanishing set. These derivatives can also be computed simultaneously with Maple's **Gradient** command. Finally, note the similarity with certain parts of the museum in Bilbao shown in Figure 4.

3.2 Second example: trigonometrical coefficients

Consider now the family of planes whose equation is $F(x, y, z, u) = 0$, where $F(x, y, u) = x \cos u + y \sin u + z \cos 2u + 1$ and u is a real parameter. The envelope obtained according to Definition 2 has

the following parametric presentation

$$\begin{cases} x = (2w \cos(u)^2 - 3w - 1) \cos(u) \\ y = (2w \cos(u)^2 + w - 1) \sin(u) \\ z = w \end{cases} \quad (8)$$

For every real number u , there exists a real number v such that $\cos u = \frac{1-v^2}{1+v^2}$ and $\sin u = \frac{2v}{1+v^2}$. With this substitution, Equation (8) can be transformed into a rational parametrization, and from there polynomials are defined, which generate an ideal J in $\mathbb{R}[x, y, z, v, w]$. By elimination, an ideal JE is obtained in $\mathbb{R}[x, y, z]$, generated by a single polynomial $G(x, y, z)$ given by:

$$\begin{aligned} G(x, y, z) = & x^6 + 3x^4y^2 + 15x^4z^2 + 3x^2y^4 - 78x^2y^2z^2 + 48x^2z^4 + y^6 + 15y^4z^2 \\ & + 48y^2z^4 - 64z^6 - 18x^4z - 144x^2z^3 + 18y^4z + 144y^2z^3 - x^4 - 2x^2y^2 \\ & + 80x^2z^2 - y^4 + 80y^2z^2 + 128z^4 + 16x^2z - 16y^2z - 64z^2. \end{aligned} \quad (9)$$

The desired variety is $V(G)$. An extra factor z^2 appears in the output and is irrelevant to the question. Two views of the envelope are shown in Figure 7 in order to emphasize its remarkable features. In the first one, the cuspidal edge is enhanced. Denote by $F(x, y, z)$ the left hand side of Equation

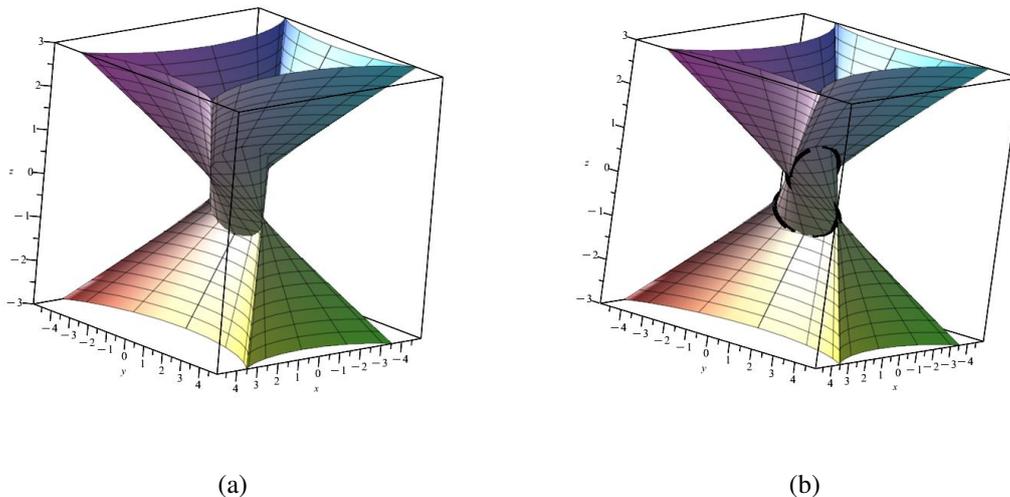


Figure 7: The envelope of the family of planes with equation $x + uy + 2u^2z - 3u^3 = 0$

(9). The singular points are found by solving the system of equations $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$. These partial derivatives can also be computed simultaneously with Maple's **Gradient** command. The **solve** command provides an answer featuring the *RootOf* place holder, which can be resolved using the **allvalues** command. The output consists of 8 real isolated points and 4 imaginary points, which are

irrelevant here, and 4 space curves, given by the following parametric presentations:

$$\begin{aligned}
 (x, y, z) &= (0, 2\sqrt{-2t^2 - 2t}, t) \\
 (x, y, z) &= (0, -2\sqrt{-2t^2 - 2t}, t) \\
 (x, y, z) &= (2\sqrt{-2t^2 + 2t}, 0, t) \\
 (x, y, z) &= (-2\sqrt{-2t^2 + 2t}, 0, t)
 \end{aligned}
 \tag{10}$$

Together, the first pair of equations describes a circle. So does the 2nd pair. The cuspidal edge is the union of two circles in perpendicular planes. Figure 8 helps to understand its structure and position in the envelope. Note that 4 more components of non-isolated singular points *seem* to exist, but they

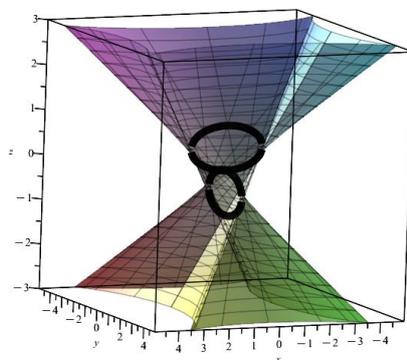


Figure 8: Enhanced visualization of the cuspidal edge

do not appear in the algebraic output. A strong zooming should show that the surface is smooth there, but this requires a more powerful DGS than what we used.

The algebraic abilities of the Computer Algebra System (CAS) in use here, namely Maple 2022, are identical for the case of plane curves and for the two cases of surfaces. We mean that the same commands, such as **solve**, can be utilized. The software uses pattern recognition and the eventual options entered by the user to decide which algorithm to use. The same remarks apply to visualization, with the **plot3d** command instead of **plot**, together with the suitable options. Note that the same **plot3d** command is useful to plot surfaces and curves in \mathbb{R}^3 , but for the last a **spacecurve** command is also available. Note also that visualization is more problematic in the 3D case than in the plane [21], and this makes the transition non-trivial. In particular, the representation on the screen (a 2-dimensional surface) of objects in 3D space requests the choice of a suitable mesh. The standard choice is rectangular, and yields often strange plots, as explained in [17]. Choosing an accurate mesh is a must, and generally a parametric presentation of the surface provides an accurate plot. This has been discussed in [43]. Figure 7 shows that quite well. Anyway, a consequence is that the respective role importance of a DGS and of a CAS are switched: in the 3D case, the CAS is dominant.

4 Envelopes of 2-parameter families of surfaces

A definition of envelopes of a 2-parameter family of surfaces, similar to Definition 2 reads as follows:

Definition 5 Let $\mathcal{S}_{u,v}$ be a family of surfaces in the 3D-space given by an equation of the form $F(x, y, z, u, v) = 0$, where u and v are real parameters. If it exists, an envelope of this family is the surface given by the solutions of the following system of equations:

$$\begin{cases} F(x, y, z, u, v) = 0 \\ \frac{\partial F}{\partial u} F(x, y, z, u, v) = 0 \\ \frac{\partial F}{\partial v} F(x, y, z, u, v) = 0 \end{cases} . \quad (11)$$

We will now explore two examples, both of families of spheres centered on an ellipsoid, but whose topology is different.

4.1 Envelope of a 2-parameter family of spheres

We consider the ellipsoid \mathcal{E} in \mathbb{R}^3 whose respective half-axes are denoted by a, b, c respectively ($a > 0, b > 0, c > 0$). A parametric presentation for the ellipsoid \mathcal{E} is as follows:

$$\begin{cases} x = a \cos u \sin v \\ y = b \sin u \sin v \\ z = c \cos v \end{cases} \quad (12)$$

For $a = b = c = 1$, \mathcal{E} is the unit sphere.

A family of spheres \mathcal{S}_k , centered on \mathcal{E} and with radius $1/k$, where $k > 0$ is determined by the implicit equation

$$(x - a \cdot \cos u \sin v)^2 + (y - b \cdot \sin u \sin v)^2 + (z - c \cdot \cos v)^2 - \frac{1}{k^2} = 0. \quad (13)$$

We denote by $G_k(x, y, z, u, v)$ the polynomial in the left-hand side of Equation (13). If the family $\{\mathcal{S}_k\}$ has an envelope, it is determined by the solutions of Equations (11) for G_k . With Maple's **solve** command, we solve Equations (11); as the output involves the *RootOf* placeholder, we need to use the **allvalues** command. Actually, in order to explore the envelopes for different values of the radius, we write a procedure in Maple¹; see [S2]:

```
with(plots):
# the given ellipsoid
srfce := plot3d([2*cos(u)*sin(v), sin(u)*sin(v), cos(v)], u = 0 .. 2*Pi,
v = 0 .. Pi, transparency = 0.5, color = red, scaling = constrained)
# the procedure
s := proc(k, a, b, c)
local G, derGu, derGv, envlp, envlp1, envlp2;
```

¹It is often possible to modify the code, to make it shorter, sometime to use loops where we do not; what is proposed can work with any version of the software.

```
G := (x - a*cos(u)*sin(v))^2 + (y - b*sin(u)*sin(v))^2
+ (z - c*cos(v))^2 - 1/k^2;
derGu := diff(G, u); derGv := diff(G, v);
envlp := allvalues(solve({G = 0, derGu = 0, derGv = 0}, {x, y, z}));
envlp1 := plot3d([rhs(envlp[1][1]), rhs(envlp[1][2]), rhs(envlp[1][3])],
    u = 0 .. 2*Pi, v = 0 .. Pi, color = yellow, transparency = 0.5);
envlp2 := plot3d([rhs(envlp[2][1]), rhs(envlp[2][2]), rhs(envlp[2][3])],
    u = 0 .. 2*Pi, v = 0 .. Pi, color = blue, transparency = 0.5);
display(sph, envlp2, envlp1);
end proc
```

Figure 9 shows the output of the procedure for $(a, b, c) = (4, 2, 1)$ and radius $r = 1/k$ equal to $2, 1, 1/2, 1/4$.

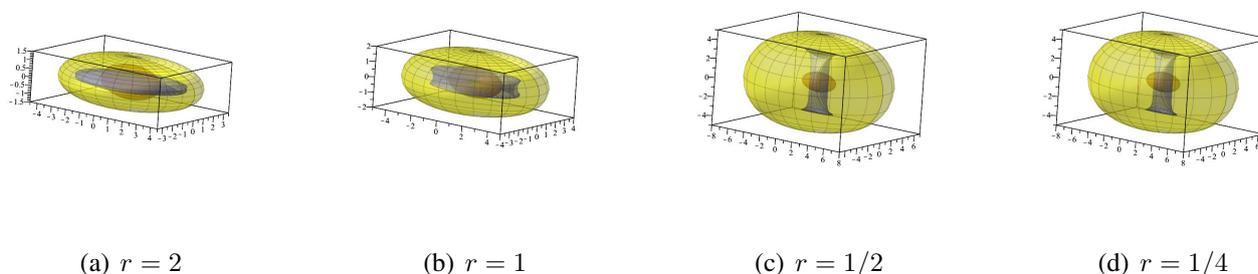


Figure 9: Envelopes of spheres centred on an ellipsoid

We should mention that some experiments have been made writing them separately; only then a general pattern appeared and enabled to write the code above, i.e. a loop. The various experiments reveal that the envelope has always two components², an external one seeming smooth, and an internal one which may have non-isolated singularities. The study of these singularities requires heavy computational machinery. Two possibilities exist:

1. If an implicit equation of the envelope has been obtained, eventual candidates for singularity are given by the vanishing of the gradient. For this, commands are available in various versions such as **Gradient** or **Nabla** (in Maple's *VectorCalculus* package they are equivalent). A non trivial drawback is that it is generally impossible to distinguish the two components by algebraic means. We mean that the equation is given by an irreducible polynomial, whose vanishing locus is the union of two distinct components. Another difficulty is that it may be impossible (at least in a reasonable time), to derive an implicit equation for the envelope. This happens with an example described later. Sometimes, when a polynomial has been found for the envelope, the singular locus can be obtained in an automated way, using the *algcurv* package and its command **singularities**.

²Actually, each component is an envelope of the given family. The discussion of the non-uniqueness of an envelope is beyond the scope of the present paper.

2. Working with the parametric presentation of the envelope, the candidates for singularity are given by the vanishing of the first derivative vector, which may be easier to find using the CAS, but here too the computations may be unilluminating. The eventual singular points seem to be non-isolated. Moreover, as shown in Figure 9, the singular locus may be the union of two components. Here too, it is impossible to distinguish the two components by algebraic manipulations.

4.2 Envelope of the family of unit spheres centered on an ellipsoid

The envelopes of families of circles centered on an ellipse have been explored in [19]. In certain cases they revealed isolated singular points, generally cusps. We explore now the family of unit spheres centered on an ellipsoid, and the influence of the parameters a, b, c of the ellipsoid on the shape of the envelope, namely on the existence of singular points. The experimental work, based on visualization, leads to conjecture that the internal component of an envelope has generally non-isolated singularities, but not always. On the one hand, this is not a surprise, as we shall encounter such a situation in Section 2 with the cuspidal edge of an envelope of a family of planes. On the other hand, the actual existence of these singularities has to be proven. For this, we use the algebraic affordances of the software.

A canonical ellipsoid \mathcal{E} , i.e. an ellipsoid having the coordinate axes as symmetry axes, is given by the following polynomial equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, c are positive real parameters, or equivalently, by the following parametrization:

$$\begin{cases} x &= a \cos u \cos v \\ y &= b \sin u \cos v \\ z &= c \sin v \end{cases} \quad (14)$$

Actually, \mathcal{E} has also a rational parametrization:

$$(x, y, z) = \left(\frac{2au}{1 + u^2 + v^2}, \frac{2bv}{1 + u^2 + v^2}, \frac{c(1 - u^2 - v^2)}{1 + u^2 + v^2} \right). \quad (15)$$

The proof is similar to the proof for a circle in the plane (see [12]). Once again, a trigonometric parametrization provides a more accurate plot than a rational one (see [43, 11]), and we use it for graphics. For the algebraic computations, a rational parametrization such as given in Equation (15) is preferred: it leads to polynomial equations, enabling the usage of Gröbner bases packages, elimination and resultants; see [10, 11, 18].

We begin the exploration using GeoGebra, as the software allows to drag points directly on the plot using the mouse. Plot the ellipsoid, and for a general point on it, plot a unit sphere centered at this point. Figure 10(a) show three of the possible unit spheres. Using the **Move point** feature with **Trace On**, we obtain the output displayed in Figure 10(b). The interaction man-and-machine, when dragging points on the ellipsoid, provides a “cloudy” picture of what can be the envelope. Because of the previous experience in 2D for envelopes of unit circles centered on an ellipse [19], we conjecture that two components exist: one external, visible on the screen, and one internal, quite invisible here.

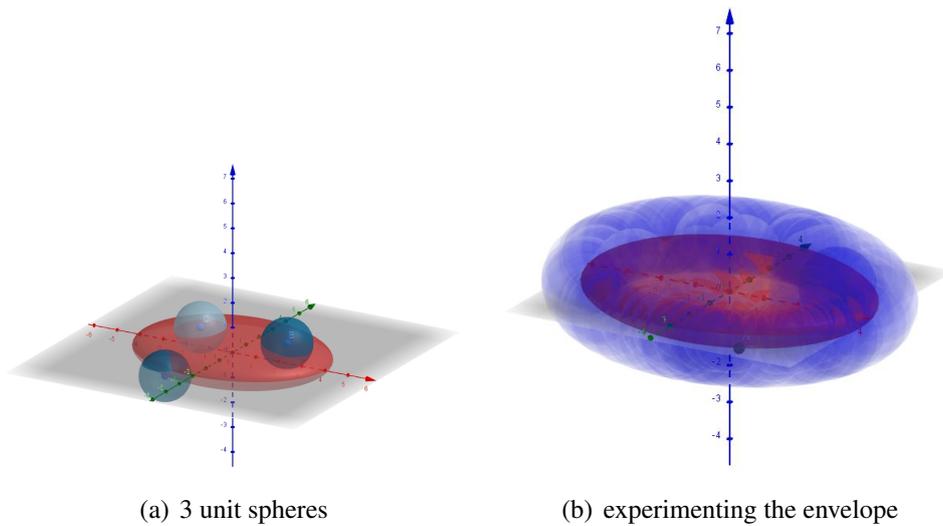


Figure 10: Dynamic exploration with GeoGebra

Even changing the transparency of the plots does not improve visibility of what happens inside the ellipsoid. Therefore, switching to precise algebraic computations and plotting with a CAS is crucial and yields a readable visualization.

With the trigonometric parametrization of the ellipsoid \mathcal{E} in Eq. (14), we obtain a 2-parameter implicit equation for the family of unit spheres centered on \mathcal{E} :

$$(x - a \cos u \cos v)^2 + (y - b \sin u \cos v)^2 + (z - c \sin v)^2 - 1 = 0. \tag{16}$$

Denote the left hand side of Equation (16) by $F(x, y, z, u, v)$. According to Definition 5, an envelope of the family of spheres, if it exists, is determined by the following system of equations:

$$\begin{cases} (x - a \cos u \cos v)^2 + (y - b \sin u \cos v)^2 + (z - c \sin v)^2 - 1 = 0 \\ (a^2 - b^2) \sin 2u \cos^2 v + 2by \cos u \cos v - 2ax \cos v \sin u = 0 \\ (a^2 - b^2) \cos^2 u \sin 2v - 2ax \cos u \sin v + (b^2 - c^2) \sin 2v - 2by \sin u \sin v + 2cz \cos v = 0 \end{cases} \tag{17}$$

Using the **solve** command, we obtain parametric presentations for two components. Denote

$$D = (((a^2 - b^2)c^2 \cos^2 u + a^2(b^2 - c^2)) \cos^2 v - a^2b^2).$$

Then a parametric presentation for one component of the envelope is:

$$\begin{cases} x = \frac{1}{D} [\cos u (ac^2((a^2 - b^2) \cos^2 u + a(b^2 - c^2)) \cos^2 v - a^3b^2 + \sqrt{-(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - b^2)b^2c}) \cos v] \\ y = \frac{1}{D} [\cos v (b((a^2 - b^2)c^2 \cos^2 u + a^2(b^2 - c^2))) \cos^2 v - a^2b^3 + c \sin u \sqrt{-a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos(v)^2 - a^2)}] \\ z = \frac{1}{D} (\sqrt{-b^2a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - a^2b^2)} + c((c^2(a^2 - b^2) \cos^2 u + a^2(b^2 - c^2) \cos^2 v - a^2b^2)) \sin v \end{cases} \tag{18}$$

and for the second component

$$\begin{cases} x = \frac{1}{D} [\cos u \cos(v) (((-a^2 + b^2)ac^2 \cos^2 u + (-b^2 + c^2)a^3) \cos^2 v - a^3b^2 \\ \quad + \sqrt{-((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)) \cos^2 v - b^2} b^2 c) \\ y = \frac{1}{D} [-b \sin u (-((a^2 - b^2)c^2 \cos^2 u + a^2b^2 - a^2c^2) \cos^2 v + a^2b^3 \\ \quad + \sqrt{-a^2(((a^2 - b^2)c^2 \cos^2 u + (b^2 - c^2)a^2) \cos^2 v - a^2b^2)} c) \cos v \\ z = \frac{1}{D} [\sin v (-\sqrt{-a^2(((a^2 - b^2)c^2 \cos^2 u + a^2(b^2 - c^2)) \cos^2 v - a^2b^2)} b^2 \\ \quad + ((c^2(a^2 - b^2) \cos^2 u + a^2(b^2 - c^2) \cos^2 v - a^2b^2) c)] \end{cases} \quad (19)$$

The algebraic differences are very small; actually as the equations are quadratic, the two components correspond to each sign of the square root of a discriminant. This explains why, by successive squaring, both parametrizations provide the same polynomials.

Figure 11 shows the case $(a, b, c) = (5, 2, 1)$ and Figure 12 shows the case $(a, b, c) = (4, 4, 2)$

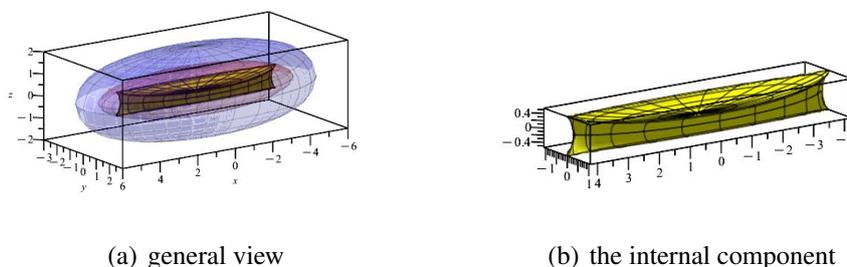


Figure 11: Envelope of unit spheres centered on an ellipsoid - $(a, b, c) = (5, 2, 1)$

In both cases, the obtained envelope is the union of two components. The external one seems to be

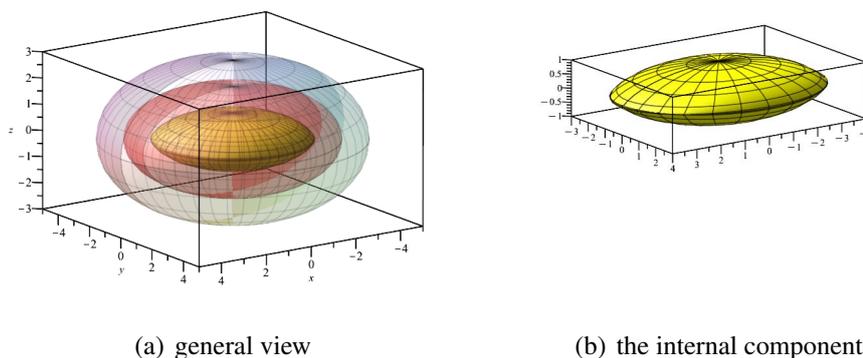


Figure 12: Envelope of unit spheres center on an ellipsoid - $(a, b, c) = (4, 4, 2)$

smooth, but the internal one seems to have singular points. It appears that, in the two examples, the

sets of singular points have different structures, whence the need for a fine study. Nevertheless, more experimental work reveals that this is not always the situation, as shown in Figure 13 for $(a, b, c) = (7, 4, 4)$.

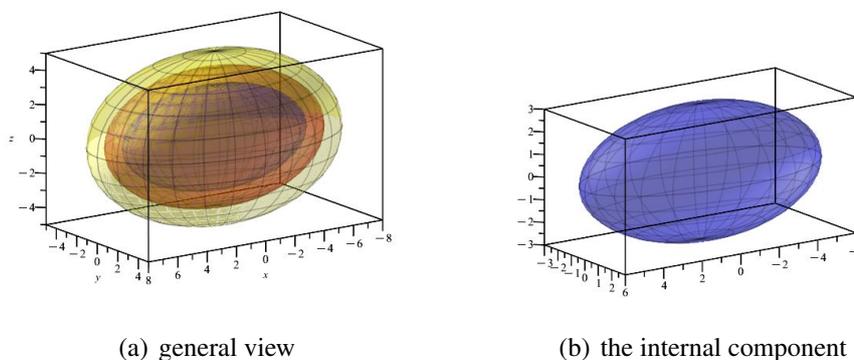


Figure 13: Envelope of unit spheres center on an ellipsoid - $(a, b, c) = (7, 4, 4)$

In order to study the singularities, we use the method described in [5], chap. 5.2. Equations (18) determine a mapping $F := \mathbb{R}^2 \rightarrow \mathbb{R}^3$. We rewrite Equations (18) in the form

$$(x, y, z) = (F_1(u, v), F_2(u, v), F_3(u, v)),$$

defining a function $F(u, v) = (F_1(u, v), F_2(u, v), F_3(u, v))$. Of course, the same method is applicable for Equations (19). The Jacobian matrix of F is

$$JF := \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} \end{pmatrix} \quad (20)$$

Let us study the case where $(a, b, c) = (4, 3, 1)$. Using the *LinearAlgebra* package and the command **Rank**, we find that generically, the rank of this matrix is equal to 2. This mapping is regular at points where its Jacobian matrix has maximal rank. The singular points are the vanishing points of the 3 minors of the matrix. The Jacobian is computed by the command **Jacobian** from Maple’s *Vector-Calculus* package. Then, in order to use the **Minor** command from the *LinearAlgebra* package, it is necessary to add a column on the right of the Jacobian matrix. This, and the subsequent computations, can be unilluminating, and we did not obtain a result in a “reasonable” time. Therefore we went in another direction.

At this point, we wish to recall a study from [19]: the envelope of a family of circles centered on an ellipse shows generally two components. The external one is smooth, and the internal one has four cusps. An example is displayed in Figure 14. The case $r = 1$ seems to have 2 smooth components; this can be easily proven. For $r = 2$, the plot suggests that there may be 2 singular points. If $r = 3$, we identify an internal component with 4 cusps. In all these cases, the algebraic computations are similar, the actual way depending on whether the curve is given by a parametrization or by an implicit equation.

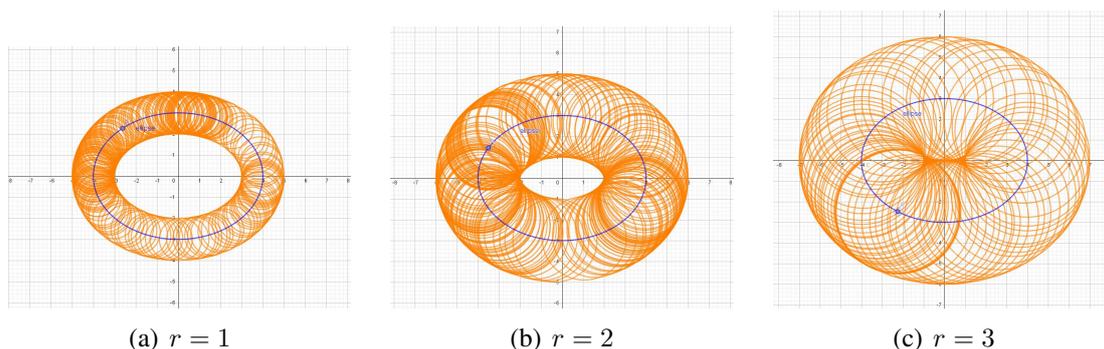


Figure 14: Families of circles centred on the same ellipse with radii 1,2,3.

For our problem here in 3D, the parametric presentation of the envelope is quite heavy. We may intersect the ellipsoid and the two components by a plane parallel to a coordinate plane, as shown in Figure 15. What remains to do is to prove that the plane curve obtained as the intersection of the plane

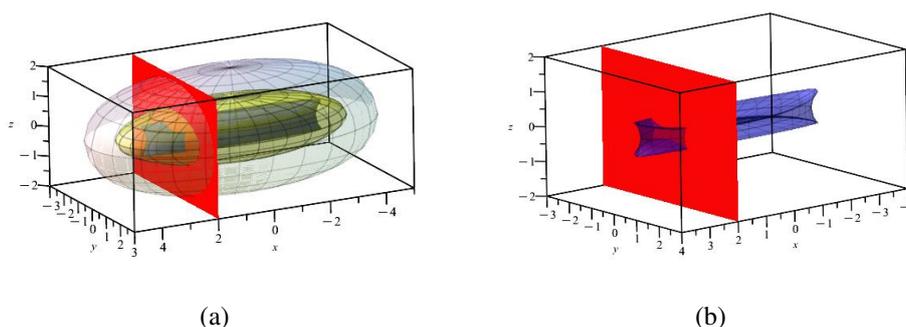


Figure 15: The intersection of a plane with the surfaces - cusps appear

with the internal component of the envelope has generally 4 cusps. This is standard work in a plane.

We said earlier that the usage of a trigonometric parametrization yields a nice output; see [43]. This is the reason why we began as above. Nevertheless, the study of singularities should be easier with an implicit equation. For this purpose, it is more efficient to use a rational parametrization of the surface (if possible), leading to polynomial expressions and enabling further use of the Gröbner package. Of course, in theory, this may be possible only with rational curves.

We start from the rational parametrization of the ellipsoid in Equation (15). We consider now the family of unit spheres centered on the ellipsoid \mathcal{E} . Their general equation is:

$$\left(x - \frac{2au}{1 + u^2 + v^2}\right)^2 + \left(y - \frac{2bv}{1 + u^2 + v^2}\right)^2 + \left(z - c \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right)^2 - 1 = 0. \quad (21)$$

Denote the left-hand side by $F(x, y, z, a, b)$. We solve now the corresponding system of equations (5). The desired envelope has 2 components. As the output is quite heavy, we do not present it here; we provide Maple code in a [companion file](#). Transforming the parametric equations into polynomials

using **expand**, **sort**, and commands from the Gröbner package with the *grlex* option, we obtain 3 polynomials in 5 variables. Here are the polynomials for $(a, b, c) = (4, 3, 1)$:

$$\begin{aligned}
 P_1(x, y, z, u, v) &= (9216 + (576v^2 + 36)x^2)u^6 + (18432v^2 - 16164 + (864v^4 + 136v^2 - 216)x^2)u^4 \\
 &\quad + (9216v^4 - 14408v^2 + 9144 + (576v^6 + 164v^4 - 376v^2 + 36)x^2)u^2 \\
 &\quad + 144(v^2 + 1)^2 \left(-\frac{1}{4} + \left(v^4 - \frac{14}{9}v^2 + 1 \right) x^2 \right) - 2304u^7x \\
 &\quad + (-6912v^2 + 1728)u^5x + (-6912v^4 + 3008v^2 + 1728)u^3x \\
 &\quad - 2304(v^2 + 1) \left(v^4 - \frac{14}{9}v^2 + 1 \right) ux + 144u^8x^2 \\
 P_2(x, y, z, u, v) &= (11520 + (1296u^2 + 144)y^2)v^6 + (23040u^2 - 18432 + (1944u^4 + 369u^2 - 360)y^2)v^4 \\
 &\quad + (11520u^4 - 20700u^2 + 11520 + (1296u^6 + 306u^4 - 846u^2 + 144)y^2)v^2 - 3888v^7y \\
 &\quad + (-11664u^2 + 2160)v^5y + (-11664u^4 + 5076u^2 + 2160)v^3y \\
 &\quad - 3888(u^2 + 1) \left(u^4 - \frac{7}{4}u^2 + 1 \right) vy + 324v^8y^2 + 324(u^2 + 1)^2 \left(u^4 - \frac{7}{4}u^2 + 1 \right) y^2 \\
 P_3(x, y, z, u, v) &= -135 \left(u^2 + \frac{128}{135}v^2 \right) (u^2 + v^2 - 1)^2 \\
 &\quad + 72(u^2 + v^2 + 1) \left(u^4 + \left(2v^2 - \frac{7}{4} \right) u^2 + v^4 - \frac{14}{9}v^2 + 1 \right) (u^2 + v^2 - 1)z \\
 &\quad + 36(u^2 + v^2 + 1)^2 \left(u^4 + \left(2v^2 - \frac{7}{4} \right) u^2 + v^4 - \frac{14}{9}v^2 + 1 \right) z^2
 \end{aligned}$$

What remains to be done is to define an ideal $J = \langle P_1, P_2, P_3 \rangle$ in $\mathbb{R}[x, y, z, u, v]$ generated by these 3 polynomials and to eliminate the variables u, v . One method is to use the **EliminationIdeal** command. Its output is an ideal JE in $\mathbb{R}[x, y, z]$. The desired envelope is contained in the variety $V(JE)$. Actually, the variety $V(JE)$ is the Zariski closure of the envelope. Other methods exist, also implemented in various CAS. Nevertheless, at the time we write, we were unable to obtain an answer using the CAS in a reasonable time³. Indeed, implicitization is an important issue, in particular for engineers, but we explained earlier how to work without an implicit equation. Let us mention that, when needed, approximate methods have been developed, for example in [40].

4.3 A 3D version of a nephroid

Denote by \mathcal{S} the unit sphere centered at the origin. It has the following parametric presentation:

$$\begin{cases} x = \cos u \cos v \\ y = \cos u \sin v \\ z = \sin u \end{cases}, \quad u \in [0, 2\pi], \quad v \in [0, \pi]. \quad (22)$$

A sphere $\mathcal{S}_{u,v}$ centered on \mathcal{S} and tangent to the xy -plane has a radius equal to $|\sin u|$, whence an equation of the form $F(x, y, z) = 0$, where the following holds:

$$F(x, y, z) = (x - \cos u \cos v)^2 + (y - \cos u \sin v)^2 + (z - \sin u)^2 - \sin^2 u. \quad (23)$$

³Z. Kovács tried also to make this. After 12 hours, the computer crashed, having run out of memory. I thank him (not the computer) here for the effort.

According to Definition 5, an envelope of this family of spheres, if it exists, is the solution set of the following system of equations:

$$\begin{cases} (x - \cos u \cos v)^2 + (y - \cos u \sin v)^2 + (z - \sin u)^2 - \sin^2 u & = 0 \\ 2(x - \cos(u) \cos(v)) \sin(u) \cos(v) + 2(y - \cos(u) \sin(v)) \sin(u) \sin(v) \\ \quad - 2(z - \sin(u)) \cos(u) - 2 \sin(u) \cos(u) & = 0 \\ 2(x - \cos(u) \cos(v)) \cos(u) \sin(v) - 2(y - \cos(u) \sin(v)) \cos(u) \cos(v) & = 0 \end{cases} \quad (24)$$

The output of the `solve` command shows two components. The first one is given by

$$\{x = \cos(u) \cos(v), y = \cos(u) \sin(v), z = 0\} \quad (25)$$

and the 2nd one by

$$\{x = \cos(v) \cos(u) (2 \sin(u)^2 + 1), y = \cos(u) \sin(v) (2 \sin(u)^2 + 1), z = 2 \sin(u)^3\} \quad (26)$$

Equation (25) describes the unit circle on the xy -plane. This is an expected case: a sphere $\mathcal{S}_{u,v}$ whose center is on the xy -plane is actually reduced to a single point, and these centers form exactly the unit disk of this plane. Equation (26) describes a non degenerate component. Figure 16 shows, on the envelope in (a), and the envelope (more transparent), with the unit sphere and one of the spheres of the family in (b); a worksheet is available as [S3].

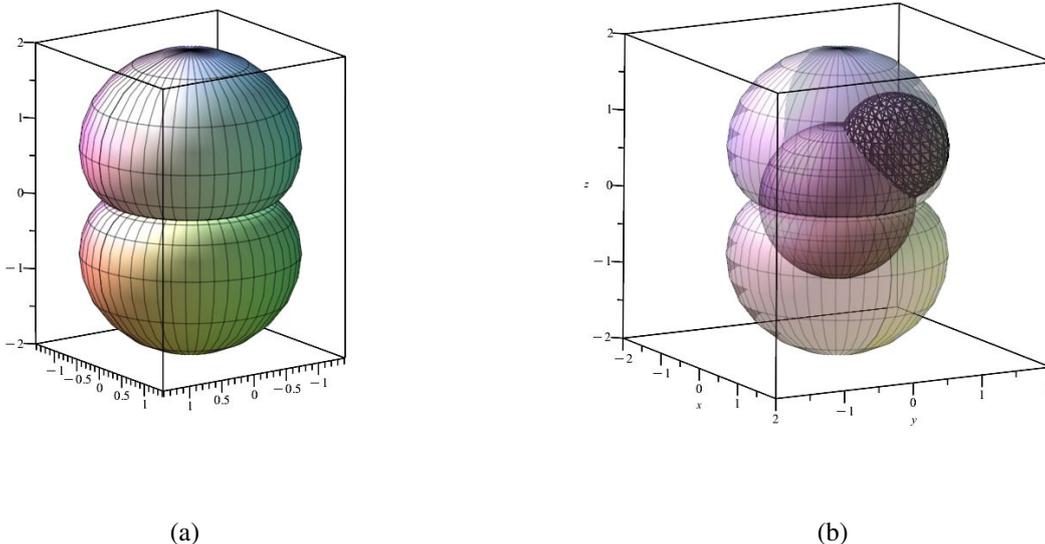


Figure 16:

Note that the sphere, element of the family, is a less accurate plot: it has been plotted using the `implicitplot3d` command, while the other surfaces have been obtained by a parametric plot. This issue has been discussed in [43]. The command `sphere` is also available and provides a parametric plot. In the present case, it is not a crucial issue.

It is well known that for every $u \in [0, 2\pi]$, there exists $s \in \mathbb{R}$ such that $\cos u = \frac{1-r^2}{1+r^2}$ and $\sin u = \frac{2r}{1+r^2}$. write similar formulas for $\cos v$ and $\sin v$ as rational functions of a real s . Using these formulas, we transform the trigonometric parametrization of the envelope (26) into a rational parametrization, which can be implicitized, by the process described earlier. The following polynomials are obtained:

$$\begin{aligned} P_1(x, y, z, r, s) &= r^6 s^2 x - r^6 s^2 + r^6 x + 3r^4 s^2 x + r^6 - 9r^4 s^2 + 3r^4 x + 3r^2 s^2 x \\ &\quad + 9r^4 + 9r^2 s^2 + 3r^2 x + s^2 x - 9r^2 + s^2 + x - 1 \\ P_2(x, y, z, r, s) &= r^6 s^2 y + 2r^6 s + r^6 y + 3r^4 s^2 y + 18r^4 s + 3r^4 y \\ &\quad + 3r^2 s^2 y - 18r^2 s + 3r^2 y + s^2 y - 2s + y \\ P_3(x, y, z, r, s) &= r^6 z + 3r^4 z - 16r^3 + 3r^2 z + z \end{aligned}$$

Denote $J_N = \langle P_1, P_2, P_3 \rangle$ the ideal that they generate in $\mathbb{R}[x, y, z, r, s]$. By elimination of the parameters r, s , we obtain an ideal JE_N in $\mathbb{R}[x, y, z]$ generated by a single polynomial. The generator of JE_N that we obtained is the following polynomial:

$$\begin{aligned} F(x, y, z) &= (z - 2)(z + 2)(2z^2 + 1)(4x^6 + 12x^4 y^2 + 12x^4 z^2 + 12x^2 y^4 + 24x^2 y^2 z^2 + 12x^2 z^4 \\ &\quad + 4y^6 + 12y^4 z^2 + 12y^2 z^4 + 4z^6 - 12x^4 - 24x^2 y^2 - 24x^2 z^2 - 12y^4 - 24y^2 z^2 \\ &\quad - 12z^4 + 12x^2 + 12y^2 - 15z^2 - 4) \end{aligned}$$

The envelope that we are looking for is the variety $V(F)$.

The first two factors correspond to two tangent planes to the obtained surface, which are parallel to the xy -plane, the 3rd factor has no real root, and the last factor gives an implicit equation of the surface. No surprise, it is of degree 6.

5 Discussion

The Institut Henri Poincaré in Paris has a large [collection of physical models of surfaces](#) in 3-dimensional space. These models have been built in 19th century, and not many people know about their existence. Such collections exist at other institutions around the world, but this one is really impressive. It has been also photographed, at least partially, by Man Ray. A recent description has been written by Apéry [1], accompanied by photos. Some of them are known as envelopes of specific families of surfaces.

Constructions with wood or other materials are a wonderful tool for teaching. It seems that it is not fully known how the Parisian models have been calculated and constructed. However, other techniques exist today, such as 3D printing. This modern technology requests the description of the surface by mathematical means and their implementation into mathematical software and from there into the language understood by the 3D printer. Whence the importance to acquire the needed mathematical skills, accompanied by the requested technological discourse, according to Artigue [2]. It is common sense that the usage of a Computer Algebra System has here two main components, algebraic and graphical. The first one requests mathematical knowledge, such as in described in Section 2, in particular regarding the different definitions of envelopes. The computations in the above examples have been performed according to Definitions 2 and 5, and not according to Definition 4, a 3D version of what Kock calls an impredicative definition [27]. Then the CAS is used to solve systems of

equations and simplify the output; note that among the requirements from a CAS, simplification tools, including trigonometric simplification and factorization, are very important [15]. When working with rational parametrizations of surfaces, the CAS tools for computing in polynomial ideals (Gröbner bases, elimination, etc.) are important; see [10]. The graphical part requests understanding of the difference in quality between implicit plots and parametric plots. 3D printing avoids the problem of visualisation on a screen, i.e., a copy of a plane, of an object in 3D such as a space curve or a surface, but raise other issues. Many of the educational benefits of this technology are described in [30].

In education, special attention should be given to the fact that questions may remain without a full answer. In Section 4.1, we could derive a parametric representation of the envelope under study, but were unable to obtain an implicit equation, probably because the computations were too heavy for our current system. At a certain extent, the usage of multiple technologies may help fill the gap, at least for the exploratory part of the work. After all, in education, open problems are more appealing than final answers, and the 5th C of education, namely Curiosity, may not be the least important.

Working out examples of rational curves and radii given by rational function provides a setting where polynomials can be derived and the algebraic packages of the CAS can be applied. Nevertheless, the meaning of the parameter has an influence. For example, we refer to the difference between a rational and a trigonometric parametrization of a circle. The second case gives a more uniform behaviour of the animations, and the plot is also more readable; see [19]. For more advanced examples, in particular in 3D, other choices of the mesh may be useful; Maple has about 20 different options for the choice of the mesh. But their usage depends in the user's literacy with the CAS; see also [43]. We did not use such options for the sake of the reader who may not be familiar with their usage.

In any case, visualization is the name of the game. It requests its own technological skills. The transition from 2D to 3D made dragging points less useful for exploration⁴. In general, the automated tools available in 2D are not always available in 3D. As an example, GeoGebra-Discovery has a command for certain kinds of envelopes of families of plane curves, but nor for surfaces in 3D space. Nevertheless, some of the tools that we used with Maple belong to the category of automated tools. Note that for certain computations we use Gröbner packages, whose algorithms are present in the programming of certain automated tools. The mouse is still the good tool to rotate the plot in order to have numerous viewpoints, but it is difficult to identify a point on a given surface shown on the screen. A consequence is that the respective roles of the different kinds of software, CAS and DGS, change with the transition from 2D to 3D. The dialog between CAS and DGS evoked in [18] for 2D should be improved in 3D also. This needed dialog has been already evoked in the past in [36, 37], but there is still a lot to do.

Finally, we wish to emphasize that we chose to work mostly with Definition 2, called analytic by Kock [27]. This is the easiest for computing. Thom [41] mentioned that the theory of envelopes has few theorems and numerous special cases. Therefore, 1-parameter and more than that, 2-parameter families of surfaces may reserve numerous interesting cases. We must still mention that the different definitions of an envelope correspond to different problems, safety zones, robotic paths, design of pipeline networks, etc. They may be a source of inspiration for artistic work also.

⁴The interest in dragging points in the plane is described in [39].

6 Acknowledgements

The first author's work has been partially supported by the Roland and Astrid CEMJ Chair at JCT.

7 Supplementary Electronic Material

[S1] Maple worksheet for the plane with trigonometric coefficients, in Section 1.

[S2] Maple worksheet for the envelopes of spheres centered on an ellipsoid, in subsection 4.1.

[S3] Maple worksheet for a 3D nephroid, in subsection 4.3.

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