

A STEAM approach to canal surfaces

Thierry Dana-Picard

ndp@jct.ac.il

Department of Mathematics
Jerusalem College of Technology
Havaad Haleumi Street 21
9116011 Jerusalem, Israel

Abstract

Canal surfaces are a special occurrence of envelopes of parameterized families of surfaces in 3D space, namely envelopes of families of spheres centered on a given space curve. When the radius of the sphere is constant, they are called tubular surfaces, or simply tubes. In every country, numerous examples can be found all around (water pipes, canals for traffic of merchandises, etc. All these can incite to study such objects or models in mathematics. Dynamic Geometry Systems are an efficient tool to model such concrete objects, as they enable both a geometric construction and an exploration of the corresponding algebraic data (equations, parametric presentation). In order to have a more powerful tool for the algebraic computations, a Computer Algebra System can be used, whence a new need to communicate between DGS and CAS. In the past, a concrete construction of such canal surfaces was viewed as an artistic creation, whose abstract basis was quite complicated to grasp. For the beginner, understanding the switch from a concrete occurrence towards abstraction (equations, etc.) was a non-trivial issue. Neither was the reverse direction. We start from a geometric description of a spinal curve and of spheres centered on it, from there we derive equations whose solutions describe the desired surface. Nowadays, these paths can be completed by a new step, from an abstract presentation with equations towards a concrete realization using 3D printing. This approach, from real world to mathematical presentation, and then back to concretization, is typical of STEAM education.

1 Introduction

In the past, mathematical domains used to be considered separately, both in research and in teaching. Along the generations, more and more connections between domains emerged, producing richer domains of knowledge. Bridges have been built between mathematics and other domains of knowledge; a modern approach is called STEAM, an acrostic for Science, Technology, Engineering, Arts and Mathematics. This approach involves exploration, inquiry, discovery, critical thinking and creativity.

Envelopes of parametric families of plane curves and of surfaces in 3D space are a live domain of study. They have numerous applications in industry, such as robotics, zones of safety [13], depollution of soils [22]. Despite their importance, they almost disappeared from the syllabus already in the sixties of the 20th century, and Thom complained about that situation [32]. New technologies enabled to revive the topic, and to propose it to both pre-service and in-service teachers. The first software used in [17] was Derive to explore 1-parameter families of plane curves. In subsequent papers, another Computer Algebra System (CAS) and a Dynamic Geometry System (DGS) have been used. In [13, 18, 19], these are Maple and GeoGebra. They are used in the present paper too.

A central feature of a DGS is the possibility to visualize and to manipulate directly on the screen all the objects of plane geometry. The main tools are point dragging and slider bars. Both are commanded by the mouse, and are central for the man-and-machine relationship. In GeoGebra a slider bar can run an automatic animation also. The most basic object in GeoGebra are points; for example a line is defined by 2 points, and a circle can be defined with a center and a point on the circle. Some of the points are free, and further constructs depend on them. Dragging a free point using the mouse induces automatically changes in all the constructs dependent on this point. Similar features characterize slider bars, and multiple sliders can be defined for a family of objects. In order to understand an object in 3D shown on a screen, the possibility to turn around using the mouse is a must, in order to discover the object from different points of view. This grants the possibility to explore, analyze, discover, and then to conjecture results which have to be proven later. Proofs can be written in a "traditional" way, we mean by pen-and-paper, but a DGS offers also tools for automated exploration and automated proofs. We refer especially to the algorithms developed by Wu [33, 34, 35] and to new algorithms which are continuously developed since then, these developments being applied in classrooms and labs, and receiving feedback from them [29, 5, 4, 9, 10, 24, 28]. Recio and Vélez analyze the different work areas allowed by a DGS: "While automatic proving deals with verifying geometric statements, automatic discovery tries to find complementary hypotheses for statements to become true or to find the missing hypotheses so that a given conclusion follows from a given incomplete set of hypotheses".

Computer Algebra Systems have been initially developed around symbolic computations, whence their name, *to do mathematics*. Quickly other features have been implemented. We use the possibilities of symbolic computations together with the graphical affordances, to *teach/learn mathematics*. We mean to explore, conjecture, develop a more profound understanding and, if possible, prove. DGS contribute strongly to this trend.

In contradiction to other scientific domains where the objects under study are graspable physically, and are often "on the table", a mathematical object is an abstract one and can be grasped only via different representations, symbolic, numerical, graphical, etc. [21]. Switching between registers of representation enables to consider the same object from different points of view. Generally, a CAS offers the switching from either symbolic or numerical representation to graphical representation. A DGS offers this switch, but also a switch in reversed direction, and knows how to translate the changes made when dragging points or using a slider directly in the algebraic representation. In GeoGebra, the algebraic window and the graphical windows are fully synchronized. This too is important for R & D and classroom work.

The graphical affordances of the two kinds of software are sometimes different. In our study, animations are important. GeoGebra enables continuous animations using the **Animation on** of a slider, and also animation driven by the user's hand on the mouse. A CAS requires writing a specific

command such as Maple's **animate** and then to press a button to run the animation. Several options are available, such as speed, number of frames, parameter increment, etc. Both kinds of software, DGS and CAS, allow to save the animation as an animated GIF. In [24], Kovács writes that "animations can be performed quickly enough by using today's computers with implicit computer algebra use. In the background, without notifying the user on the fact that thousands of symbolic computations are performed, a pure dynamic geometry experience can be relived, making geometry more geometrical than before." More advanced features, described in [28], are available as a companion package called *GeoGebra discovery*, new versions being published at a frequent pace¹.

We must mention that among other commands for automated exploration and discovery, GeoGebra has a command **Envelope** for the automated determination of envelopes in a 2D setting. The current version of **Envelope** requests the tracer to be a variable point and not a slider for variations of a real parameter; see [6, 7, 8]. This has an influence on our choices: here we use a parametric representation of a space curve, and not a pure geometric construction. This has been already the case for the study of a Maltese Cross as an envelope in [13].

Finally, we add here an old-new way of representing mathematical objects: 3D construction. The evolution of teaching passed from the concrete representation in wood or metal towards more abstract drawing on a sheet of paper or plots on a screen. We are now back to a genuine 3D represen

2 Enhancing motivation via concrete occurrences

2.1 Canals in real life

Canals are almost ubiquitous in the world:

- Small canals to bring water from the source to the consumers. Figure 1(a) shows an ancient canal in Peru² - pay attention that the shape of the section varies, to ease the fluidity of the flow. Note that most of these canals are open above, but not always, and pipes can be found. Figure 1(b) shows an underground part of the aqueduct from Solomon Pools in Judea mountains to Jerusalem. Figure 2 shows an exit to an open-air part of the same system, dating hundreds of



(a) Precolombian water canal



(b) Aqueduct in Judea

Figure 1: Ancient canals

¹The package is freely downloadable from <https://github.com/kovzol/geogebra-discovery>

²Source: <https://www.audleytravel.com/ca/peru/country-guides/inca-sites-to-visit>

years earlier, and an early 20th century replacement by a pipe³. Note that the pipe seems to have variable radius, despite the fact that the inner part may be quite different, the larger radius corresponding to connections of different sections (needed in particular to connect straight and curved parts).

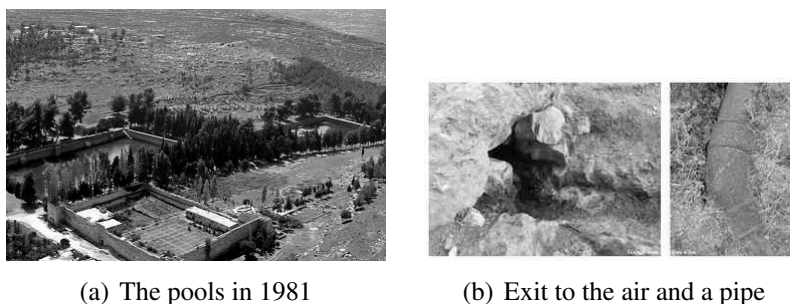


Figure 2: Solomon's Pools

- Larger canals for the circulation of goods. In Europe, heavy loads used to be transported by waterways, nowadays these canals are often used for tourism and cruises (Figure 3).

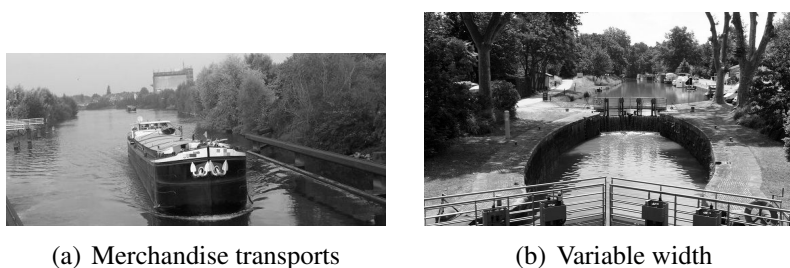


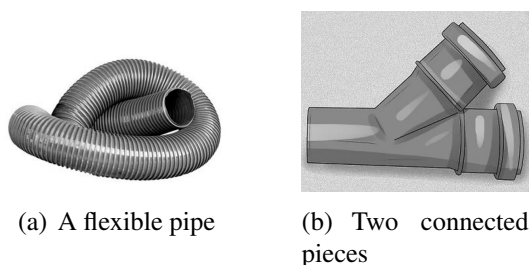
Figure 3: Circulation of goods and tourism

- Entertainment parks yield nice examples. Figure 4 shows an open water slide and a tubular one.
- At a smaller scale, Figure 5 show pipes which can be found in any household, for a washing machine or a dish washing machine. One example has variable radius (periodic when the pipe is straight); as by nature it is very flexible, it can help to visualize several mathematical objects. It is quite clear that for the second example, two different mathematical surfaces have to be defined and "glued" together.
- Washers are shown in Advanced Calculus as examples of surfaces of revolution. They can be viewed also as canal surfaces.

³Sources: (a) https://commons.wikimedia.org/wiki/File:Solomon%27s_pools2.jpg; (b) <https://www.jpost.com/travel/around-israel/solomon-pools-fit-for-a-king>.



Figure 4: two kinds of water slides



(a) A flexible pipe

(b) Two connected pieces

Figure 5: Home water pipes

These different examples may be an incentive for students for the mathematical study that we propose. As discussed in [14, 16], the usage of items from the cultural background of the students encourage them to study of various more theoretical topics, and then to discover applications. Moreover, visits on the field (mathematical trails) can contribute to enhancing motivation. The configurations may be very different, but the presence of such canals in the vicinity of the students may be such an incentive to study mathematical models, a part of a mathematical trail. The variety itself, or the the changes along the canals will be a good reason to study canal surfaces with variable radius.

2.2 Mathematics and Art

The “Institut Henri Poincaré” in Paris has a large collection of models of surfaces in 3-dimensional space, established in the 19th century. Some of them are models of canal surfaces. Apéry wrote recently a description accompanied by photos [1]. Figure 6 shows two examples, both from the Brill-Schilling collection: (a) a ruled surface⁴, and (b) a canal surface, which is the envelope of spheres with variable radius⁵. This is the topic of next section. Note that explicit equations are not always available, and that even then, it is understood that the endeavor to construct these objects was quite hard.

Figure 7(a) shows a piece of string art⁶, and Figure 7(b) presents a GeoGebra model⁷. This model

⁴<https://patrimoine.ihp.fr/item/3490>

⁵<https://patrimoine.ihp.fr/item/3392>

⁶It illustrates activities in France, by EchoSciences-Occitanie - <https://www.echosciences-sud.fr/communautes/diffusion-de-la-culture-mathematique-en-occitanie/articles/mathematiques-des-fils-tendus-vers-le-reve>

⁷<https://www.geogebra.org/m/nkxrwmxd>

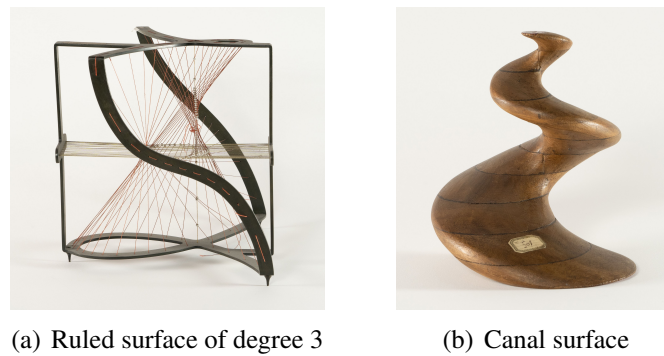


Figure 6: Physical models of surfaces

is 2-dimensional, but the threads create a ruled surface, because of their thickness. This topic may be explored in a course in Linear Algebra or elsewhere, and the visualization is important. Moreover, the animation helps to visualize the envelopes of the various segments in motion.

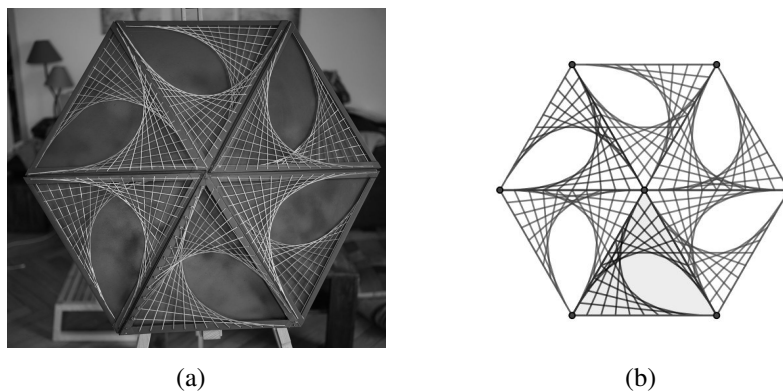


Figure 7: A piece of string art and a computerized model

The empty zones, shaped as leaves, are bounded by something which can be modeled as an envelope of segments. The description using equations may be unilluminating, thus we will not describe it here, and we address simpler examples in the next sections.

3 Envelopes and canal surfaces

There exist 4 different definitions of envelopes of 1-parameter families of plane curves, given in [11]; all of them can be adapted to 1-parameter families of surfaces in 3-dimensional space. A version for 1-parameter families of surfaces of Definition 5.8 in [11] (page 80) is the following:

Definition 1 Let S_k denote a family of 1-parameter surfaces in 3D space.

1. The characteristic curves of the family are the limit curves, when ϵ goes to 0, of the intersection of nearby surfaces S_k and $S_{k+\epsilon}$.

2. The envelope of this family, if it exists, is the union of the characteristic curves.

Following a 3D version of [11] (5.12 page 81), it can be shown that \mathcal{E} is a surface tangent to every \mathcal{S}_k .

Suppose that the surfaces \mathcal{S}_k are given by an equation of the form $F(x, y, z, k) = 0$, where k is a real parameter. The intersection $\mathcal{S}_k \cap \mathcal{S}_{k+\epsilon}$ is determined by the following system of equations:

$$\begin{cases} F(x, y, z, k) = 0 \\ F(x, y, z, k + \epsilon) = 0 \end{cases} \quad (1)$$

After sidewise subtraction and computation of a limit for ϵ going to 0, we obtain the following result:

Proposition 2 *If it exists, an envelope of this family is the solution set of the system of equations:*

$$\begin{cases} F(x, y, z, k) = 0 \\ \frac{\partial}{\partial k} F(x, y, z, k) = 0 \end{cases} \quad (2)$$

Note that this is the only definition of an envelope given by Berger [3]. For elementary situations (such as [18] III, example 2), the system (2) of (generally) non-linear equations can be solved by hand. In heavier situations, a Computer Algebra System can be used (the user has already some understanding of what happens and the usage is not purely as a black box. The **solve** command may use different algorithms, according to whether the equations are polynomial or not. The interested reader can refer also to [17, 19]. As we will see in the next subsection, for canal surfaces we consider families of spheres, and the characteristic curves are circles.

3.1 Canal surfaces

Definition 3 *Let \mathcal{C} be a curve in the 3-dimensional space given by a parametrization $M(t)$ and let r be a function which associates a positive real number $r(M)$ to every point on \mathcal{C} . For a point M on \mathcal{C} , we denote by S_M the sphere centered at M with radius $r(M)$. If it exists, the envelope of the family of spheres S_M is called a canal surface. The curve \mathcal{C} is called the spinal curve of the canal surface. If the function r is constant, we call the envelope a pipe surface or a tubular surface, or simply a tube.*

With the emergence of various technologies, at the beginning plotters, later CAS and Dynamic Geometry Systems, it became possible to model envelopes in general, and canal surfaces in particular, on a screen. As these plots are projections of 3D objects on a 2D screen, they have some drawbacks. For example, Figure 8 illustrates the envelope of spheres of radius $3/2$ centered on the parabola in the x, y -plane whose equation is $y = x^2$. We use here the synchronisation between the 2D window and the xy -plane in the 3D window. The dotted curves in the 3D window are the circles of the 2D window. From Figure 8(a) it can be understood that the envelope is self-intersecting, which is not clear from Figure 8(b). We have here a typical situation of the difficulty to visualize 3D objects on a 2D screen. Figure 8(c) is nicer and shows the envelope, but the self-intersection is hidden. Conviction (not a proof) can be obtained when revolving the plot on the screen. Creating the model with a 3D printer allows the learner to turn the surface with his/her fingers; see Section 5. Conviction becomes stronger.

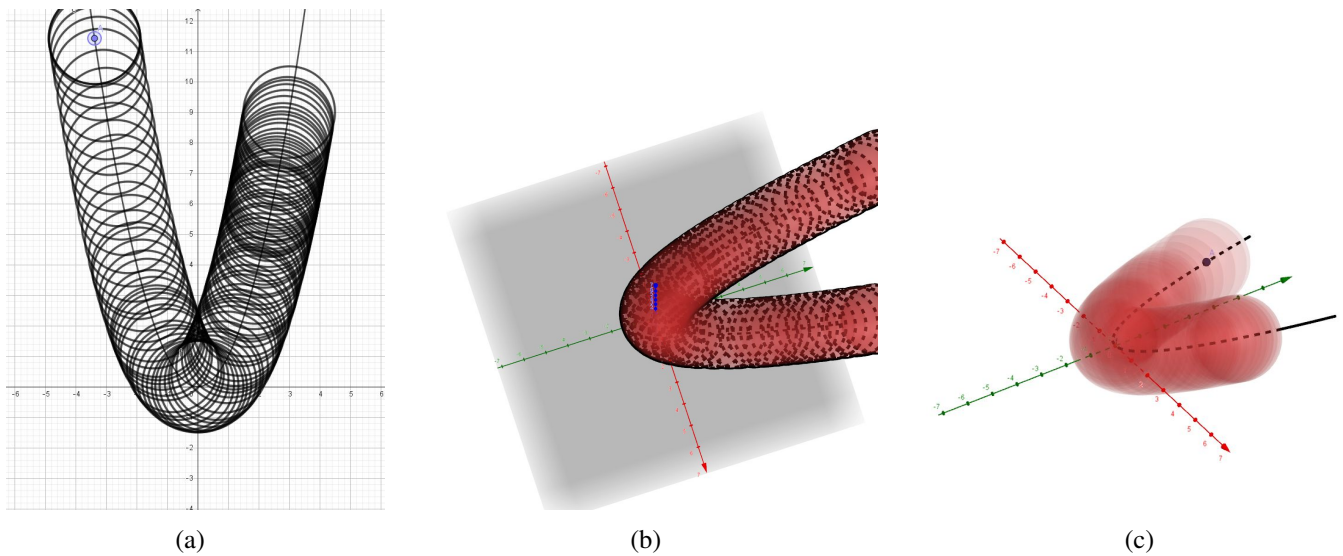


Figure 8: A self-intersecting envelope

3.2 Example: a torus

We consider the family of spheres of radius $1/2$ centered on the circle in the xy -plane whose center is at the origin and with radius equal to 2 (the spinal curve). A general equation of these spheres is

$$(x - 2 \cos t)^2 + (y - 2 \sin t)^2 + z^2 - \frac{1}{4} = 0. \quad (3)$$

We denote by $F(x, y, z, t)$ the right-hand side of Equation (3). Then we have:

$$\frac{\partial F}{\partial t} = 4(x - 2 \cos t) \sin t - 4(y - 2 \sin t) \cos t.$$

Solving System (2) for this case, we obtain the two following parametric presentations:

$$\begin{cases} x = v \cot t \\ y = v \\ z = \frac{\sqrt{-15 \sin^2 t + 16v \sin t - 4v^2}}{2 \sin t} \end{cases} \quad \text{and} \quad \begin{cases} x = v \cot t \\ y = v \\ z = -\frac{\sqrt{-15 \sin^2 t + 16v \sin t - 4v^2}}{2 \sin t} \end{cases} \quad (4)$$

Note that we obtained the envelope as the union of two components, who are symmetric about the xy -plane.

Using the method described in [13, 15], we transform these parametrizations into rational parametrizations, then into a system of polynomial equations. These polynomials generate an ideal in $\mathbb{R}[x, y, z, t]$. By elimination we obtain a single polynomial in $F_1(x, y, z)$ whose vanishing locus is the desired envelope, i.e. we found an implicit equation of the envelope. In the present case, it is easy to identify this equation as an equation of a torus. Figure 9 has been obtained with GeoGebra, moving the sphere with the Trace On option. The spinal curve appears by transparency (the dotted circle)⁸.

⁸The full synchronisation between the 2D window and the xy -plane in then 3D window is very helpful for moving teh center of the sphere along the spinal curve.

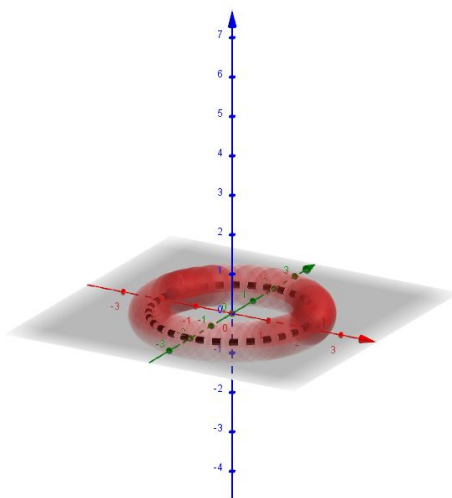


Figure 9: A torus as an envelope of spheres

3.3 Example: a necklace

Consider every point $P(r, 0, 0)$ on the x -axis as the center of a sphere of radius $|\sin r|$. They are defined in GeoGebra using the command $Sphere(A,abs(\sin(r)))$. The envelope of this family of spheres, is displayed in Figure 10; it looks like a necklace.

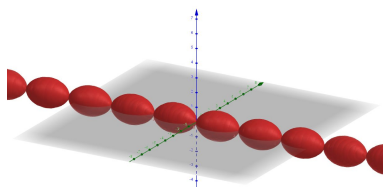


Figure 10: A necklace

The general equation of the spheres is

$$(x - r)^2 + y^2 + z^2 - \sin^2 r = 0. \tag{5}$$

Solving System (2), we obtain the two following parametric presentations:

$$\begin{cases} x = x = r - \sin r \cos r \\ y = \sqrt{\sin^4 r - t^2} \\ z = t \end{cases} \quad \text{and} \quad \begin{cases} x = x = r - \sin r \cos r \\ y = -\sqrt{\sin^4 r - t^2} \\ z = t \end{cases} \tag{6}$$

Consider now the family of circles centered on the same segment as above, with the same variable radius. Figure 11 has been obtained using the command $Circle(A,abs(\sin(r)),i)$ and **Trace On**, where i denotes a unit vector on the x -axis. The figure on the left has been obtained after one run of the

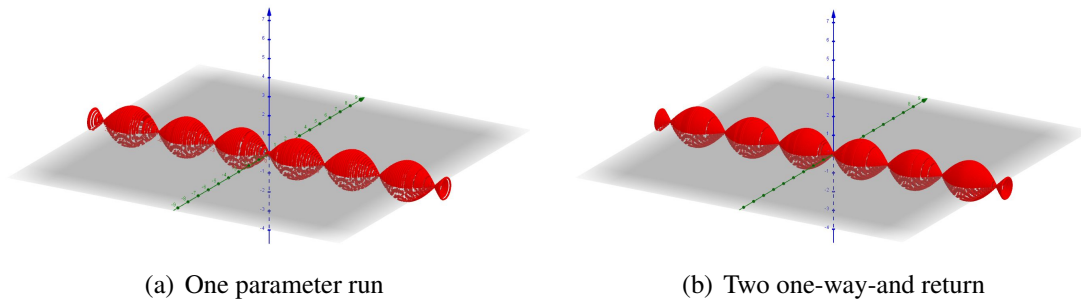


Figure 11: Another construction of the necklace

parameter, the figure on the right after 2 returns. Note the accuracy differences between them and between them and Figure 10: in Figure 11, white spots appear, which are not present in Figure 10.

Definition 2 is not applicable here, as we have a 1-parameter family of space curves and not 1-parameter family of surfaces. The space curves are defined by a parametric representation and not by an implicit equation, which is requested by Definition 2. Kock relates shortly to this case in [23] but without computational expressions. Here we can use the definition and its translation into a theorem given by the Mathcurve site⁹.

Definition 4 *The envelope of a 1-parameter family of space curves C_k is the geometric locus \mathcal{E} of the characteristic points of the curves C_k , which are the limit points of the intersection points between C_k and $C_{k'}$ when k' tends to k . These points only exist if the curve C_k is secant with the infinitely close curves C_{k+dk} . \mathcal{E} is then tangent at each of its points to a curve C_k and, in general, any curve C_k is tangent at least at one point to \mathcal{E} .*

Theorem 5 *Suppose that the curve C_k is given by*

$$\begin{cases} F(x, y, z, k) = 0 \\ G(x, y, z, k) = 0 \end{cases} \quad (7)$$

Then the envelope exists if and only if the following system of 4 equations

$$\begin{cases} F(x, y, z, k) = 0 \\ G(x, y, z, k) = 0 \\ \frac{\partial}{\partial k} F(x, y, z, k) = 0 \\ \frac{\partial}{\partial k} G(x, y, z, k) = 0 \end{cases} \quad (8)$$

has a solution for (x, y, z) for all values of k . This solution gives the parametrization of the envelope.

Note that it is rather exceptional that a family of space curves has an envelope. It is generally a curve, but may degenerate to a set of isolated points (see [17] 4.2, for an example in 2D).

Here we have $F(x, y, z, k) = (x - r)^2 + y^2 + z^2 - \sin^2 r$ and $G(x, y, z, r) = x - r$. System (8) has no real solution, therefore no envelope exists in the sense of Theorem 5. An extension of the theory would be welcome; it is discussed, for example, in [31] but is beyond the scope of the present paper.

⁹<https://mathcurve.com/courbes3d.gb/enveloppe/enveloppe.shtml>

Nevertheless, we provide the source code for an animation with Maple, showing successive positions of the circles. The animation with **Trace On** creates a surface, generated by the circles with variable radius. The number of plotted circles is determined either by the slider or by the speed of moves with the mouse; the less circles are plotted, the more discrete is the plot. In order to have a feeling of continuity, it is necessary to increase the number of frames and the number of circles which have to remain on the screen.

```
F := (x - r)^2 + y^2 + z^2 - sin(r)^2;
G := x - r;
solve({F = 0, G = 0}, {x, y, z});
circ := allvalues(%):
a1 := animate(spacecurve, [[rhs(circ[1][1]), rhs(circ[1][2]),
    rhs(circ[1][3])], z = -3 .. 3], r = -10 .. 10,
    scaling = constrained, thickness = 2, trace = 100,
    frames = 100):
a2 := animate(spacecurve, [[rhs(circ[2][1]), rhs(circ[2][2]),
    rhs(circ[2][3])], z = -3 .. 3], r = -10 .. 10,
    scaling = constrained, thickness = 2, trace = 100,
    frames = 100):
display(a1, a2);
```

Figure 12 shows a partial plot in the middle of the animation and the final plot.

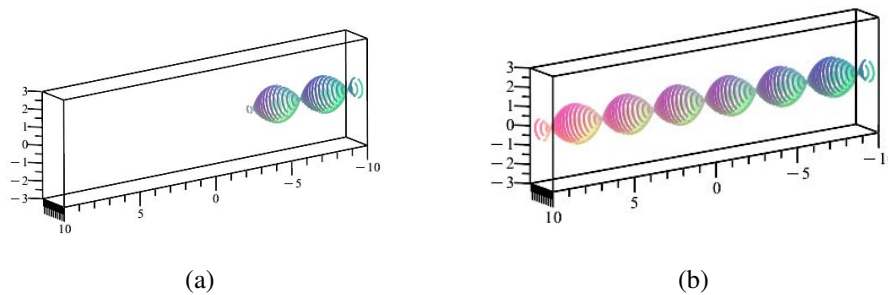


Figure 12: The necklace with a Maple animation

4 A trefoil knot as a canal surface

Consider the space curve \mathcal{C} defined by the parametric presentation

$$\begin{cases} x = \cos t + 2 \cos 2t \\ y = \sin t - 2 \sin 2t \\ z = \sin 3t \end{cases} \quad (9)$$

Two views are displayed in Figure 13, obtained with Maple. We will take this curve as a spinal curve

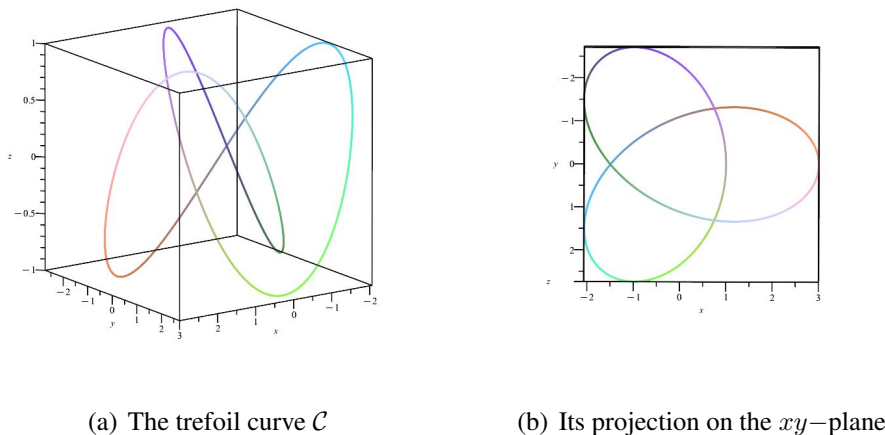


Figure 13: A trefoil space curve

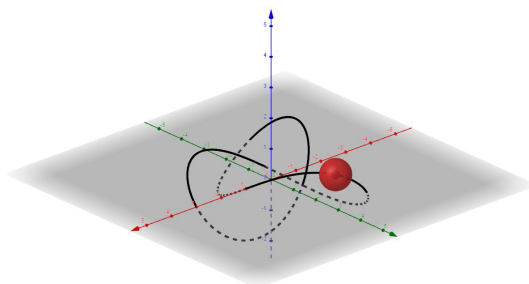


Figure 14: A trefoil space curve

for a family of spheres, namely the family of spheres centered on this curve, with radius $1/2$. Figure 14 shows the space curve \mathcal{C} and one of the spheres of the family. Figure 15 shows two views of the envelope of the family, which is a canal surface. Actually it is a tubular surface, as the radius of the spheres is constant. Both figures have been obtained with GeoGebra; for Figure 15, a point A has been attached to the curve \mathcal{C} and a sphere of center A and radius $1/2$ plotted, then A has been dragged along the curve while the **Trace On** property has been chosen for the sphere.

A general equation for spheres centered on \mathcal{C} with radius $1/2$ is

$$(x - (\cos t + 2 \cos 2t))^2 + (y - (\sin t - 2 \sin 2t))^2 + (z - \sin 3t)^2 - \frac{1}{4} = 0. \quad (10)$$

As usual, we denote by $F(x, y, z, t)$ the left-hand side in Equation (10) and solve the System of Equations (2) in this case. The output is too heavy to be displayed here, and has to be studied using the software. For the reader's convenience, we give here the source code with Maple:

```
c := spacecurve([cos(t) + 2*cos(2*t), sin(t) - 2*sin(2*t), -sin(3*t)],
                t = 0 .. 2*Pi, labels = [x, y, z]);
F := (x + (-cos(t) - 2*cos(2*t)))^2 + (y + (-sin(t) + 2*sin(2*t)))^2
```

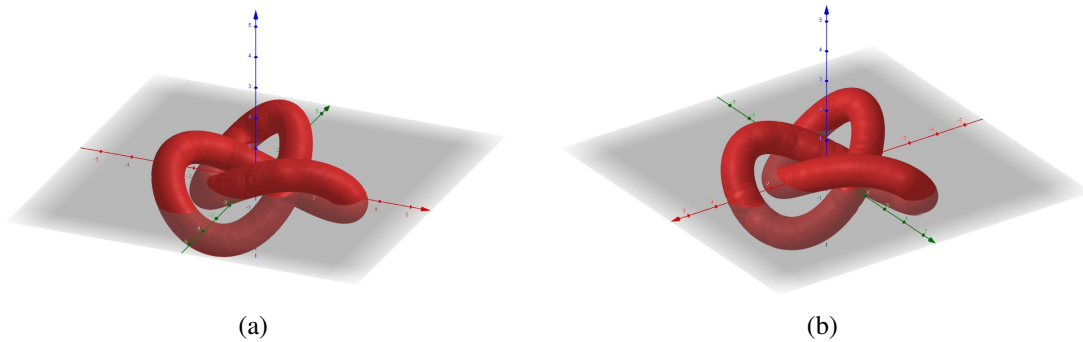


Figure 15: The concrete knot

```

+ (z + sin(3*t))^2 - 1/4;
derF := diff(F, t);
env := solve({F = 0, derF = 0}, {x, y, z});
env := allvalues(%);
x1 := simplify(rhs(env[1][1]));
y1 := simplify(rhs(env[1][2]));
z1 := simplify(rhs(env[1][3]));
s1 := plot3d([x1, y1, z1], t = 0 .. 2*Pi, y = -3.5 .. 3.5,
  numpoints = 25000, labels = [x, y, z], axes = none);
x2 := simplify(rhs(env[2][1]));
y2 := simplify(rhs(env[2][2]));
z2 := simplify(rhs(env[2][3]));
s2 := plot3d([x2, y2, z2], t = 0 .. 2*Pi, y = -3.5 .. 3.5,
  numpoints = 40000, labels = [x, y, z],
  style = patchnogrid, axes = none);
display(s1, s2);

```

Figure 16 shows the output of the **display** command. We used the *style=patchnogrid* option for one of the components only, in order to show how the mesh has been chosen by the software .

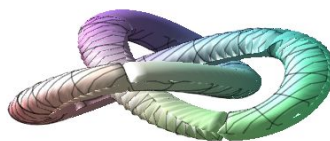


Figure 16: The trefoil knot

The output of the computation described with the source code above is really heavy (therefore we did not display it here). It cannot be reasonably used to 3D print the trefoil. Other parametrizations may be used. For example, see the method proposed on the dedicated [wikipedia page](#). There a substitution $u = \cos t$ and $v = \sin t$ is used. In order to take into account that $u^2 + v^2 = 1$, the computations have to be performed according to $v = \pm\sqrt{1 - u^2}$. Each case provides two components,

and the envelope is the union of 4 components. Here too, our computations yielded a heavy answer. A more applicable way for 3D printing has been developed with GeoGebra by Diego Lieban¹⁰.

As a final remark in this section, we should note that when plotting a curve with a DGS, the user has the choice of the plot thickness. Figure 17 shows the same trefoil knot as Figure 14, but with the maximal thickness offered by GeoGebra. Actually, this can be viewed as a view of a pipe surface, and a beginner cannot be aware of this fact.

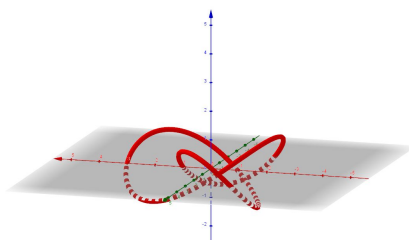


Figure 17: A pipe surface obtained by thickening the plot of a space curve

5 Discussion

5.1 New registers of representation

As mentioned previously, Duval [21] noted that a mathematical object is an abstract object, which cannot be grasped in our hands, contrary to objects of other scientific domains. It can be studied only via different registers of representation. Traditionally, these are numerical, symbolic, and graphical, and the possibility to switch between different registers is of the utmost importance. Different kinds of software provide different possibilities to switch. Generally, a CAS offers switching from symbolic representation to graphical representation. Actually it uses in between a numerical representation, but does not display it if not requested. And if requested, the numerical and graphical representations are obtained with two distinct commands. Working with a DGS (here GeoGebra), both switching directions are offered: from symbolic/numerical to graphical and from graphical to symbolic/numerical. Moreover, the dragging in a DGS enables to explore interactively, and, in the case of Figure 15, it enabled to have a very smooth display.

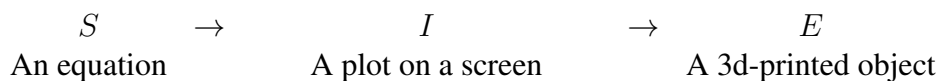
The evolution of teaching passed from the concrete representation in wood or metal (we refer here to Subsection 2.2) towards more abstract drawing on a sheet of paper or plots on a screen. In a more general framework than Mathematics Education, Jerome Bruner [12] proposed three modes of representation: Enactive representation (action-based), Iconic representation (image-based) and Symbolic representation (language-based). He described the first years of children development as

$$E \rightarrow I \rightarrow S.$$

Of course, the topic of the present paper does not belong to these first years, but we can describe

¹⁰<http://www.geogebra.org/3d/kvzcezcp>.

it in Bruner’s terminology. In our case we may have the following sequence:



The last step (marked by E) can be made concrete with 3D printing. Figure 18 shows: (a) a graphical presentation with GeoGebra, using point dragging of the center of the spheres along the curve, and (b) the result of 3D printing for the canal surfaces obtained by spheres centered on a non planar curve, given by the parametrization $(x, y, z) = (\cos 2t, \sin 2t, 3 \sin t)$.



Figure 18: A 3D printed tubular surface

This new kind of representation is sometimes feasible directly from the DGS, but in the present case it requested an extra step. Either point dragging (or using the slider bar) with **Trace On** for the spheres, or using the **Locus** command shows something which is not recognized by the software as a 3D object (a surface). Therefore some algebraic work had to be performed before the GeoGebra file could be translated into the language of the 3D printer. The 3D printer requires a specific input, mostly a triangulated mesh of the object to be printed. This has to be generated either from the equations or from the parametrization. We have here a new example of the necessity of a dialog between different kinds of mathematical software, as discussed in [20]. Here we have 3 kinds of technology involved: a CAS, a DGS and a 3D printer.

Note that we do not have much information on how the handcrafted objects presented in Subsection 2.2 (Figure 6) have been constructed.

5.2 The role of technology

Using a Computer Algebra System (CAS), it is generally possible to compute parametric presentations of these surfaces. If the original parametrization of the spinal curve is rational, the equations describing the envelope are transformed into polynomials; this has been done, for example, in [20]. Then a package for computations of Gröbner bases enables to derive an implicit equation. This is not always possible. Even when possible, it may be possible to plot with Maple, but not with GeoGebra. In both cases, there are limitations: with Maple, the plot options have to be chosen in order to avoid a strange plot (a problem analyzed in [37]), and with GeoGebra the limitation is sometimes on the degree of the polynomial.

In early 2000's a great mathematician published an interview in a French scientific magazine aimed at a general audience, where he regretted that math educators work with their students in a way too different from the way researchers work¹¹. His concern is met in the papers quoted in Section 1: working in a technology rich environment provides experimental approaches, similar for researchers and their students. Moreover, such an environment enables a revival of classical, and sometimes almost forgotten domains. In the previous studies we obtained another byproduct: a unifying framework for a family of plane curves, which had been generally viewed as different objects without any connection between them [13]. We have also an opportunity to re-express the wish to see in the near future more advances towards automated networking CAS and DGS. The history of such a wish goes back to the early 2000's and knew continuous advances, both in programming and in applications (e.g. [29, 25, 28, 20, 30]).

Three years later, Papadopoulos and Dagdilelis wrote in [26] that a "DGS allows students to generate assumptions that later can be verified or rejected through formal proof", adding that "this verification of assumptions takes place much more easily in the DGS environment than in other computational environment or in the more traditional setting of paper and pencil". We hope that the different parts of our study will contribute to an incentive to develop skills for exploration, discovery, conjecture. An accurate verification may be out of reach from the DGS alone (even when a CAS is implemented in it), and the derivation of accurate results is obtained by the networking of a DGS and a powerful CAS together, thus answering both the educators' requirements and Bourguignon's wish. This networking exists already since the implementation of the Giac CAS into GeoGebra [25]. Other important developments can be found in GeoGebra-discovery.

Combining in class mathematical software together with the concrete realizations and applications in real world is a genuine STEAM approach to canal surfaces.

Acknowledgement: The author wishes to thank Eva Ulbrich, from Johannes Kepler University, Linz, Austria, for 3D printing for him various objects. In particular Figure 18 shows her work. He wishes to thank also the anonymous reviewers for their fruitful remarks.

References

- [1] Apéry, F. *The Cabinet de mathématiques of the Henri Poincaré Institute in Paris*, Kwartalnik Historii Nauki I Tech 65 (3) (2020), 97-108. Available: <https://www.ejournals.eu/pliki/art/17658/> (accessed July 2022)
- [2] Balka, D. *Estándares de práctica matemática y STEM*, Boletín informativo del conector Math-science (2011), 6-8.
- [3] Berger, M.: *Geometry*, Springer Verlag, 1994.
- [4] Blazek, J. and Pech, P.: *Locus Computation in Dynamic Geometry Environment*, Mathematics in Computer Science 13 (1-2) (2019), 31-40
- [5] Botana, F. and Abánades, M.A. *Automatic deduction in (dynamic) geometry: Loci computation*, Computational Geometry 47(1) (2014), 75-89.

¹¹The issue of the magazine is not available anymore, therefore we did not mention the name of this mathematician

- [6] Botana, F. and Recio, T. *Some issues on the automatic computation of plane envelopes in interactive environments*, Mathematics and Computers in Simulation **125** (2016), 115-125.
- [7] Botana, F. and Recio, T. *Computing envelopes in dynamic geometry environments*, AMAI (Annals of Mathematics and Artificial Intelligence) **80**(1) (2017), 3-20.
- [8] Botana, F. and Recio, T. *A proposal for the automatic computation of envelopes of families of plane curves*, Journal of Systems Science and Complexity **32**(1) (2019), 150-157.
- [9] Botana, F., Valcarce, J.: *A dynamic-symbolic interface for geometric theorem discovery*, Computers & Education **38**, 21-35 (2002).
- [10] Botana, F. and Valcarce, J.L.: *Automatic determination of envelopes and other derived curves within a graphic environment*, Mathematics and Computers in Simulation **67** (2004), 3-13.
- [11] Bruce, J.W. and Giblin, P.J.: *Curves and Singularities*, Cambridge University Press, 1992. Online <https://doi.org/10.1017/CBO9781139172615> (2012).
- [12] Bruner, J. S. *Toward a Theory of Instruction*. Cambridge: Harvard University Press, 1966.
- [13] Dana-Picard, Th. *Safety zone in an entertainment park: Envelopes, offsets and a new construction of a Maltese Cross*, Electronic Proceedings of the Asian Conference on Technology in Mathematics ACTM 2020; Mathematics and Technology, 2020, ISSN 1940-4204 (online version).
- [14] Dana-Picard, Th., and Hershkovitz, S. *STEAM Education: technological skills, students' cultural background and Covid-19 crisis*, Open Education Studies **2**(1) (2020), 171-179.
- [15] Dana-Picard, Th. *Envelopes and offsets of two algebraic plane curves: exploration of their similarities and differences*, Mathematics in Computer Science **15**, Springer (2021).
- [16] Dana-Picard, Th., Hershkovitz, S., Lavicza, Z. and Fenyvesi, K.: *Introducing Golden Section in the Mathematics Class to Develop Critical Thinking from the STEAM perspective*, the South-East Asia Journal of STEM Education **2**(1) (2021), 151-169.
- [17] Dana-Picard, Th., and Zehavi, N. *Revival of a classical topic in Differential Geometry: the exploration of envelopes in a computerized environment*, International Journal of Mathematical Education in Science and Technology **47**(6) (2016), 938-959.
- [18] Dana-Picard, Th. and Zehavi, N. *Automated Study of Envelopes transition from 1-parameter to 2-parameter families of surfaces*, The Electronic Journal of Mathematics and Technology **11**(3) (2017), 147-160. Printed version in the Journal of Research of Mathematics and Technology **6**(2), (2017), 11-2.
- [19] Dana-Picard, Th. and Zehavi, N. *Automated study of envelopes: The transition from 2D to 3D*, The Electronic Journal of Mathematics and Technology **13**(2) (2019), 121-135.
- [20] Dana-Picard, Th. and Kovács, Z. *Networking of technologies: a dialog between CAS and DGS*, The Electronic Journal of Mathematics and Technology **14**(1) (2021), 43-59.

- [21] Duval, R. *Understanding the Mathematical Way of Thinking – The Registers of Semiotic Representations*, Springer Cham, 2017. DOI: <https://doi.org/10.1007/978-3-319-56910-9>
- [22] Zeitoun, D. and Dana-Picard, Th. *Delineation of the Zone of Influence of Pumping Wells using CAS and DGS*, the Electronic Proceedings of the Asian Conference on Technology in Mathematics ACTM 2021, Mathematics and Technology, 2022.
- [23] Kock, A. *Envelopes – Notion and Definiteness*, Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) **48**(2) (2007), 345-350.
- [24] Kovács, Z. *Achievements and Challenges in Automatic Locus and Envelope Animations in Dynamic Geometry*, Mathematics in Computer Science **13** (2019), 131-141.
- [25] Kovács, Z., Parisse, B. *Giac and GeoGebra – improved Gröbner basis computations*. In Gutierrez, J., Schicho, J., Weimann, M. (eds.), Computer Algebra and Polynomials, Lecture Notes in Computer Science 8942. Springer, 2015, 126-138.
- [26] Papadopoulos, I. and Dagdilelis, V. *Computer as a Tool of Verification in a Geometry Problem-Solving Context*, in (Novotná, J. ed) Proceedings of the International Symposium on Elementary Mathematics Teaching (SEMT), Prague 2005, 260-268. <https://www.semt.cz/proceedings/proceedings.html>
- [27] Recio, T. and Vélez, M.P. *Automatic discovery of theorems in elementary geometry*, Journal of Automated Reasoning **23** (1999), 63-82.
- [28] Kovács, Z., Recio, T., Richard, P., Van Vaerenbergh, S. and Pilar Vélez, M. *Towards an ecosystem for computer-supported geometric reasoning*, International Journal of Mathematical Education in Science and Technology **53**(7) (2022), 1701-1710, DOI: [10.1080/0020739X.2020.1837400](https://doi.org/10.1080/0020739X.2020.1837400)
- [29] Roanes-Lozano, E., Roanes-Macías, E. and Vilar-Mena, M. *A Bridge Between Dynamic Geometry and Computer Algebra*, Mathematical and Computer Modelling **37** (2003), 1005-1028.
- [30] Roanes-Lozano, E. *Boosting the Geometrical Possibilities of Dynamic Geometry Systems and Computer Algebra Systems through Cooperation*, In: M. Borovcnik, H. Kautschitsch (Eds.) Technology in Mathematics Teaching. Proceedings of ICTMT-5, Schriftenreihe Didaktik der Mathematik **25**. öbv & hpt, Vienna (2002), 335-348.
- [31] Shin, H. and Cho, S.K. *Directional offset of a 3D curve*, SMA '02: Proceedings of the 7th ACM symposium on Solid modelling and applications, 329-335. DOI: <https://doi.org/10.1145/566282.566.329>
- [32] Thom, R. *Sur la théorie des enveloppes*. Journal de Mathématiques Pures et Appliquées **XLI**(2) (1962), 177-192.
- [33] Wu, W.T. *Some recent advances in mechanical theorem proving of geometries*, in: W.W. Bledsoe, D.W. Loveland (Eds.), *Automated Theorem Proving: After 25 Years*, in: Contemporary Mathematics **29**, AMS (1984), 235-241.

- [34] Wu, W.T. *Basic principles of mechanical theorem proving in geometries*, Journal of Automated Reasoning **2** (1986), 221-252.
- [35] Wu, W.T. *Mechanical Theorem Proving in Geometries*, Springer, 1994.
- [36] MacTutor History of Mathematics Archive: *Interactive Curves: Talbot's Curve*, retrieved January 2021 from <https://mathshistory.st-andrews.ac.uk/Curves/Talbots/curvesapplet/>
- [37] Zeitoun, D. and Dana-Picard, Th.: *Accurate visualization of graphs of functions of two real variables*, International Journal of Computational and Mathematical Sciences **4**(1), 1-11, 2010..