Using Computer Multimedia to Dissolve Cognitive Conflicts of Mathematical Proof

Chun-Yi Lee & Ming-Puu Chen
lii@ice.ntnu.edu.tw, mpchen@ntnu.edu.tw
Graduate Institute of Information and Computer Education
National Taiwan Normal University
Taiwan

Abstract: This paper exemplified the design of an instructional support by means of computer-multimedia to dissolve students' cognitive conflicts during the process of a mathematical proof problem, *the pasture problem*. Cognitive conflicts occur when there are expectations which are not fulfilled. *The pasture problem* provides cognitive conflicts and encourages students to explore and bridge the gap between conjectures and proofs. In the beginning of the problem-solving process, students made a conjecture concerning the solution of *the pasture problem* and proposed a reason why it was true. The reason was often rooted in common sense or based on previous learning. Through multimedia-represented cognitive conflicts, students are more capable of visualizing and accepting the new conjecture. Then, it led them to construct a new explanation for this new conjecture naturally. It is believed that through the multimedia-supported exploration students were guided to use deductive reasoning, construct reasons to support the new conjecture, and be motivated to solve *the pasture problem*.

1. Background and Introduction

Proof is the concrete base of a house built of mathematics. For mathematicians, proof has been considered as a tool for verifying mathematical statements and explaining the reasons that support these statements [1]. Through logic and inductive reasoning, proof provides students with other learning opportunities to enhance their mathematical understanding from a rigorous perspective and helps them to see mathematics as the result of human endeavor and as a logically constructed discipline rather than a series of unrelated esoteric theorems and rules. Proof is not merely to support conviction, to respond to a distrustful nature of self-double, nor to be done as part of an obsessive ritual. Proof serves to provide explanation. Therefore teaching proof is a common activity in mathematics classroom for high school students, which is unique and different from other sciences teaching.

Although Euclidean geometry and geometric proof once occupied a central place in mathematics education, the attempts to teach them were generally not successful in the past [2]. Research indicates that many students now fail to understand the purpose of mathematical proof, are unable to construct proofs, and readily base their conviction on empirical evidence or the authority of a textbook or teacher [3]. It was very common for students to stop at the stage at which they had found a formula, and they do not feel the necessity of producing proofs. Students constructed a proof just because the teacher asked them to do so [4]. Even worse, many people think of proof as a part of geometry rather than a general mathematical process. For that reason, the great challenge is to find places within the mathematics curriculum where the need for proof is evident and the means of implementing it are within the students' mathematical ability [5].

School mathematics focuses more on the product than the process and fails to convey how mathematics has evolved as a logical system founded on proof. In this paper, proof will be considered from an alternative perspective, viewing proof construction as a problem-solving task [6]. In most proof situations, there are a lot of valid inferences that one could draw, but only a small number of these inferences will be useful in constructing a proof [7]. Under the proper circumstances, engaging students in a problem-solving task can foster the development of deep mathematical insight, useful representations for reasoning about complex mathematical concepts, and powerful problem-solving heuristics [8]. Consequently, focusing on the problem-solving aspects of proof provides insight into some important themes that other perspectives on proof do not address [6], including heuristics that mathematicians use to construct proofs [9], reasons that students come to a standstill in proof where they do not know how to go forward [10] [7], and techniques for teaching students cognitive or metacognitive strategies to overcome their difficulties related to proof [11].

The promotion of proof as a problem solving process through which mathematical knowledge and understanding have been constructed will not necessarily motivate students, though, unless they believe that they are participating in meaningful mathematical discovery. Movshovitz-Hadar [12], as well as Dreyfus and Hadas [13] identified the role of cognitive conflicts in supporting the teaching of proof. They argued that students' appreciation of the possible functions of proof can be achieved by activities in various learning environments in which the empirical investigations lead to unexpected and contradictory situations. Goldernberg, Cuoco, and Mark [14] also stated that proof, especially for beginners, might need to be motivated by a need for an explanation of why a statement is true. When proof construction is too obvious, students would likely consider that it is ritualistic and empty, and they could not recognize the purpose of proof.

Calculators, computer software tools, and other technologies assist in the collection, recording, organization, and analysis of data. They also enhance computational power and provide convenient, accurate, and dynamic drawing, graphing, and computational tools. With such devices, students can extend the range and quality of their mathematical investigations and encounter mathematical ideas in more realistic settings. It was often mentioned that the computer hinders the development of a problematic need for proof. It is the context in which the computer is a part of the teaching and learning arrangement that strongly influence the ways in which the need for proof does - or does not - arise [1]. In the context of a well-articulated mathematics program, technology increases both the scope of the mathematical content and the range of the problem situations that are within students' reach. Powerful tools for computation, construction, and visual representation offer students access to mathematical content and contexts that would otherwise be too complex for them to explore. Using the technological tools to work in interesting problem contexts can facilitate students' achievement of a variety of higher-order learning outcomes, such as reflection, reasoning, problem posing, problem solving and decision making [1] [15] [16] [17].

In the light of such evidence, what is the rationale for including proof in school mathematics, and how can proof be made more accessible to students? Today's challenge, then, is to design tasks where students experience a genuine cognitive need for conviction and where proving offers them the satisfaction of understanding why their conjectures are true [18]. In this paper, a problem-solving activity that raises students' consciousness of cognitive conflicts between conjectures and findings through technology support will be described. This activity uses computer multimedia to create the setting and atmosphere from which the contradictions arose and left the findings unresolved. Through the using of computer multimedia, students could understand the problem more clearly, clarify their misconceptions, and dissolve cognitive conflicts of mathematical proof. Therefore, students can explain and prove the findings through self-exploration naturally [1].

2. Teaching Framework

Garofalo and Lester [19] identified four problem-solving phases together with the metacognitive behaviors they engendered in performing a mathematical task: *orientation*, *organization*, *execution*, and *verification*. In the first phase, *orientation*, students are involved in assessing and understanding the problem. Metacognitive behaviors during this phase include comprehension strategies, analysis of information and conditions, assessment of familiarity with the task, initial and subsequent representations, and assessment of the level of difficulty and the chance of success. In the second phase, *organization*, students engage in planning behaviors and monitoring actions. Metacognitive behaviors during this phase consist of identification of goals and sub-goals, global planning and the local planning necessary to carry out global plans. In the third phase, *execution*, students are involved in the regulation of behavior to conform to plans. Metacognitive behaviors during this phase include performance of local actions, monitoring progress of local and global plans, and trade-off decisions such as speed versus accuracy.

In the final phase, *verification*, students engage in evaluation of decisions and results of executed plans. Metacognitive behaviors during this phase encompass evaluating decisions and checking computations. This framework provides a useful context for analysis of students' performance in mathematical problem solving processes [20].

Clark and Mayer [21] argued that success in problem-solving relies on: (1) cognitive skills---the facts, concepts, and procedures unique to a skill field. (2) meta-skills---the ability to plan, monitor, and access actions associated with problem-solving. (3) motivation---an investment of effort to persist and solve the problem. For the reason, they offered four guidelines to apply when using computer multimedia to support problems on the job: (1) Use real job contexts to build work-specific problem solving skills. (2) Provide expert models of problem-solving actions and thoughts. (3) Promote learner awareness of their problem-solving actions and thoughts. (4) Base the lesson on a detailed analysis of job expert problem-solving processes.

According to Clark and Mayer's suggestions, and Garofalo and Lester's problem solving phases, it is concluded that in *orientation* stage, computer multimedia should build the context of realistic problem-solving situation to help students to become interested in the problem and understand the problem. In *organization* stage and *execution* stage, examples of expert problem-solving actions and thinking should be provided by means of computer-multimedia in order to promote learners' awareness of and reflection on their problem-solving process. In *verification* stage, multimedia should offer the power of computation, construction, and visual representation to help students evaluate decisions and check computations. The main points mentioned above are summarized in Table 1.

Table 1 Problem Solving Teaching Framework with Multimedia Support.

Problem Solving Phase	Metacognitive Behaviors	Computer Multimedia Support
Orientation	1. Reading/ rereading 2.Initial/subsequent representations 3. Analysis of information and conditions 4. Assessment of problem difficulty	Build the context of realistic problem-solving situation.
Organization	1. Identifying goals and sub-goals 2. Making a global plan 3. Implementing a global plan 4. Drawing diagrams and organizing data into other formats	1. Provide examples of expert problem-solving actions and thinking. 2. Promote learner awareness of and reflection on their problem-solving process by making learners document their plans and by showing maps of student and expert problem-solving paths. 3. Assignments to perform activities on worked examples of expert problem-solving. 4. Assignments to write out problem-solving plans. 5. Visualizations of learners' problem-solving paths, which can be compared with the paths of experts.
Execution	1. Performance of local goals 2. Monitoring progress of local and global goals 3. Performing calculations 4. Redirecting efforts	
Verification	1. Evaluating decisions 2. Checking computations	Offer the power of computation, construction, and visual representation to help students evaluating decisions and checking computations.

3. The Pasture Problem

Many students arriving at university level are even still do not realize that fitting a formula to a pattern is not the same thing as proving it. Helping students bridge the gap from the conjecture to a proof and making them feel the need of proof are important issues in a well-designed mathematics curriculum. Cognitive conflicts and computer multimedia can provide just the new medium we need for teaching proof viewed as a problem-solving task. For this to be successful, however, we need a bank of good examples. The rest of this paper is devoted to one such, and the example of teaching the pasture problem will be described and presented based on Problem Solving Teaching Framework with Multimedia Support.

The Pasture Problem: *A shepherd has a rectangular pasture with a length of 90 meters and a width of 60 meters. The shepherd wants to construct a cross street on the pasture. Here are five designs (see Fig. 1 to Fig. 5). Do the following five figures have the same leftover area of the pasture? If not, which one of them would have the maximum leftover area of the pasture?*

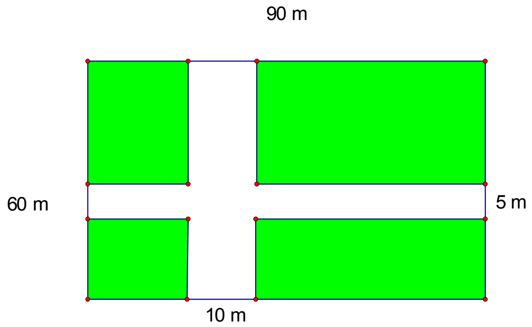


Fig. 1 Design 1

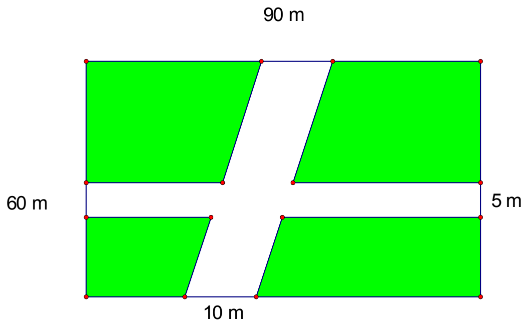


Fig. 2 Design 2

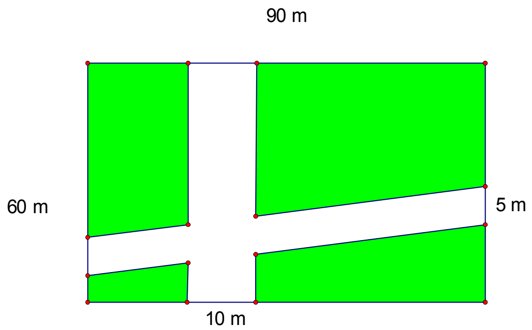


Fig. 3 Design 3

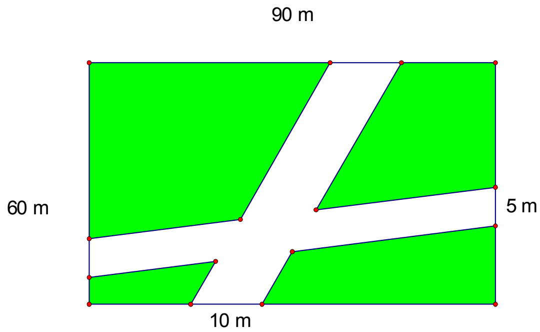


Fig. 4 Design 4

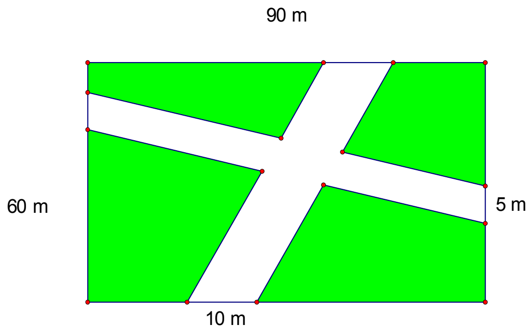


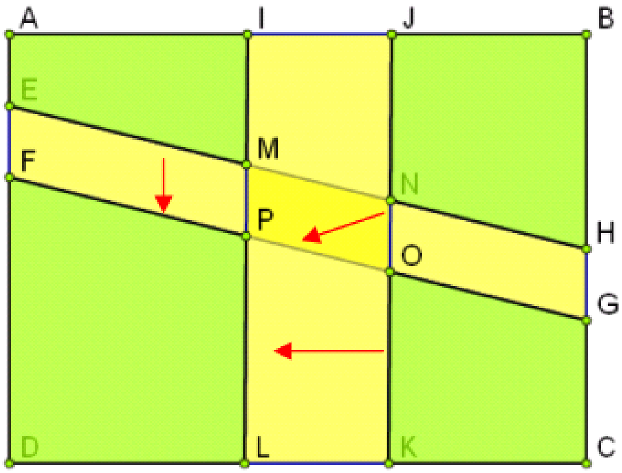
Fig. 5 Design 5

3.1 Teaching Stage 1: *Orientation*

The purpose of this teaching stage is to make students understand *the pasture problem*, including reading and rereading of the problem, initial and subsequent representations of the problem, analysis of information and conditions from the problem, and assessment of difficulty in the problem. Computer multimedia here has to create the problem situation and pose *the pasture problem* described above and then the teacher must consider students' ability to identify the problem and define it. Students will code the important elements from the problem situation. They will visualize the characteristics of the pasture problem mentally, involving relating the newly acquired information to the previously acquired information. Then the teacher gives every student a chance to guess the answer and judge the reason. Almost ninety five percent of students in the class would consider that the five figures all have the same leftover area, $4400\,m^2$. This is because they think that the four leftover pastures could combine into a large rectangle with a length of (90-10) meters and a width of (60-5) meters. This conjecture students gained plays an important role during *orientation* stage because it will lead students to generate cognitive conflicts during the next teaching stage.

3.2 Teaching Stage 2: *Organization and Execution*

The purpose of this teaching stage is to make students plan how to proceed and to execute the solution according to the plan, consisting of identifying goals and sub-goals, making and implementing a global plan, monitoring and controlling progress of a solution plan. Computer multimedia should base this lesson on a detailed analysis of job expert problem-solving processes and then provide expert models of problem-solving actions and thoughts as well as promote learner awareness of their problem-solving actions and thoughts. The teacher divided the class into eight groups. There were four students with heterogonous mathematical expertise in each group. Then the teacher provided each group the virtual manipulative [22] which could simulate *the pasture problem*, help students explore it, and guide them to form a new conjecture. Students used this tool to investigate the nature of the pasture problem, and to monitor progress of their plan of the solution (Fig. 6, Fig. 7, and Fig.8 are the displays of operations of this virtual manipulative). The teacher also had to move from group to group providing assistance via scaffolding. Through group discussion and technology support, almost every group of students found the following facts: (a) the four leftover pastures can combine into a big rectangle in the first three figures (see Fig. 6). (b) In the fourth figure, the four leftover pastures can combine into a big rectangle, but there is a small overlap of a parallelogram in the middle of the big rectangle (see Fig. 7). (c) In the final figure, the four leftover pastures can combine into a big rectangle, but there is a small gap of a parallelogram in the middle of the big rectangle (see Fig. 8).



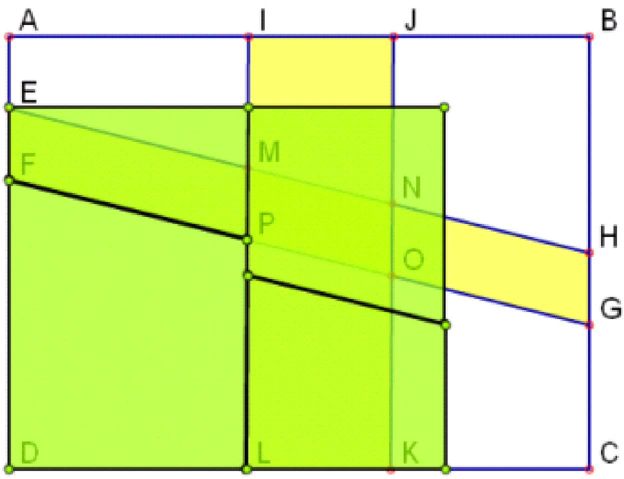


Fig. 6 The finding in condition 1

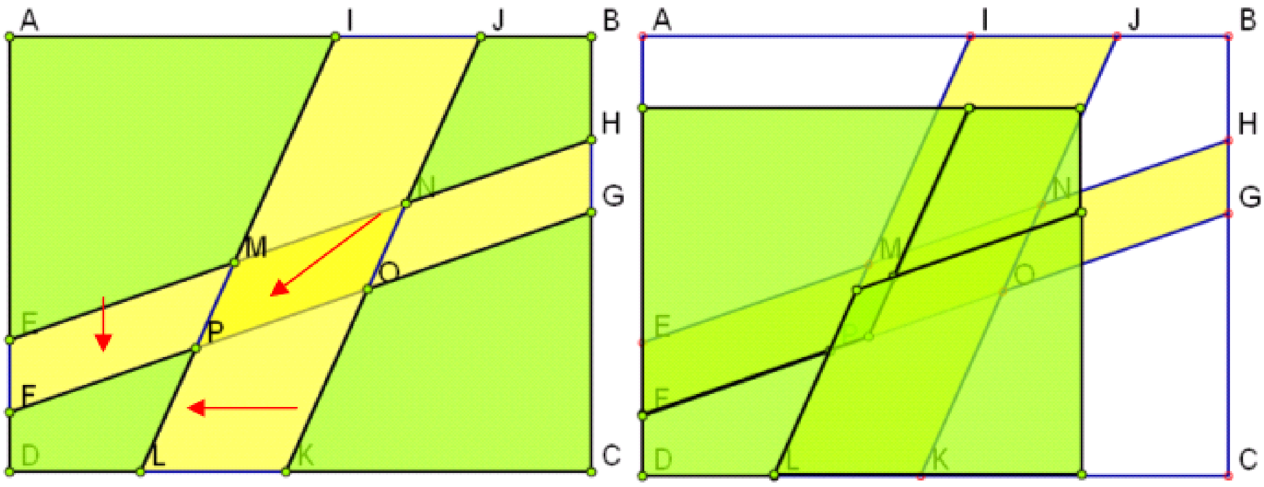


Fig. 7 The finding in condition 2

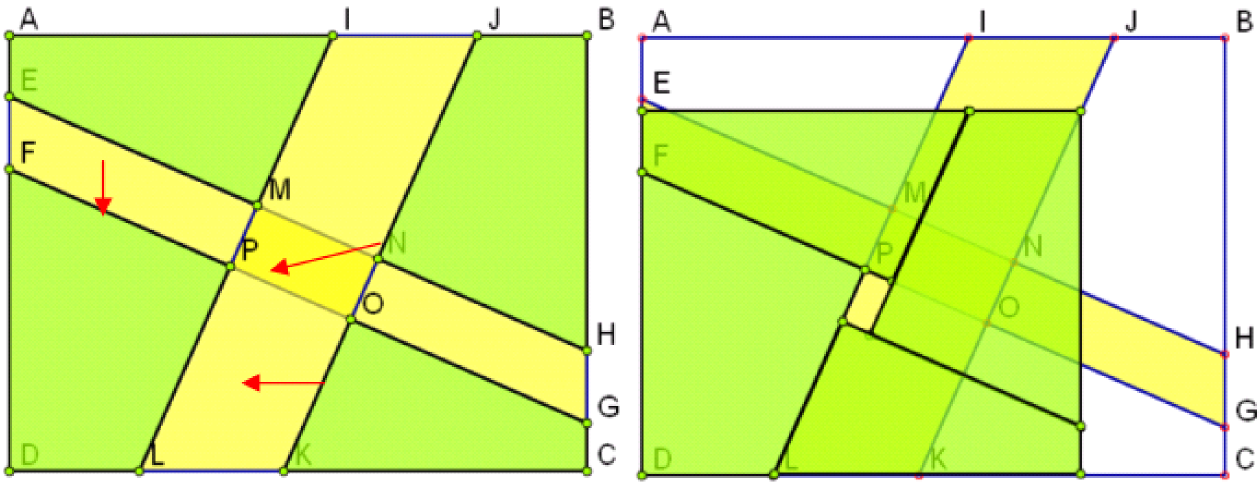


Fig. 8 The finding in condition 3

Fact (a) is an expected result whereas fact (b) and fact (c) are surprising findings. This is because students hypothesized that the four leftover pastures could combine into a big rectangle in all five figures and this intuitive belief was quite strong especially when the first three figures were checked using the virtual manipulative. But the findings of the last two figures didn't support their judgment and original conjecture. Therefore they were much surprised about the strange phenomenon occurring in their exploration via computer support. The teacher should utilize the above three facts to guide students to dissolve the contradictions. Cognitive conflicts resulting from these contradictions while checking their original conjectures might trigger a need for explanations and proofs. Students in the same group started to discuss why these surprising phenomena occur and they desired to build a mathematical model to address this issue.

3.3 Teaching Stage 3: *Verification and Proof*

The purpose of this teaching stage is to make students evaluate what they know about their performance, encompassing the interaction of a person, a solution and a strategy. Computer multimedia should play an important role in offering the power of computation, construction, and visual representation to help students verify their judgments. Because the areas of the two roads are always fixed according to the problem situation, students could get the sum of the areas for the four leftover pastures by subtracting the areas of the two roads, IJKL, EFGH from the area of rectangle ABCD, and then adding the area of the parallelogram MNOP, the intersection of the two roads (see Fig. 9). Therefore the larger the area of MNOP is, the greater the sum of the areas for the four leftover pastures is. But is the area of MNOP really different in Fact (a), Fact (b), and Fact (c)? They felt a little doubtful and did not know whether their conjecture is right. Therefore, we provided another computer tool, constructed by the Geometers' Sketchpad, to help students identify their reasoning (See Fig.9, Fig.10, and Fig.11). Students could easily find that the area of MNOP decreases from condition 1 to condition 3. And they have more confidence in their conjecture about *this pasture problem* due to the empirical evidences. Next they focused their attention on how to get the area of Parallelogram MNOP. When the teacher went around each group in the class, it is observed that students have trouble with going forward to finish this proof. There were three main difficulties that have to be overcome from the observation of their worksheets. (1) Students didn't know how to set variables in this problem. (2) Students didn't know how to express the area of Parallelogram MNOP using their set variables. (3) When they got the expression of the two set variables, students didn't know how to discuss it systematically and show their results.

At this moment, computer multimedia offered each group three powerful hints to support the process of reasoning when they didn't have any idea about how to solve this problem. *Hint 1*: Let $\angle MPO=\theta$ and $\angle PMR=\alpha$, using trigonometry to find the area of parallelogram MNOP. *Hint 2*: Find the expressions of $\overline{MP}$ and $\overline{OQ}$ from $\triangle MPR$ and $\triangle QSO$ respectively. *Hint 3*: When $\angle PMR$ is fixed, try to find the relationship between the area of MNOP and the degree of $\angle MPO$. Students could press the hint buttons to get these three hints (see Fig. 12).

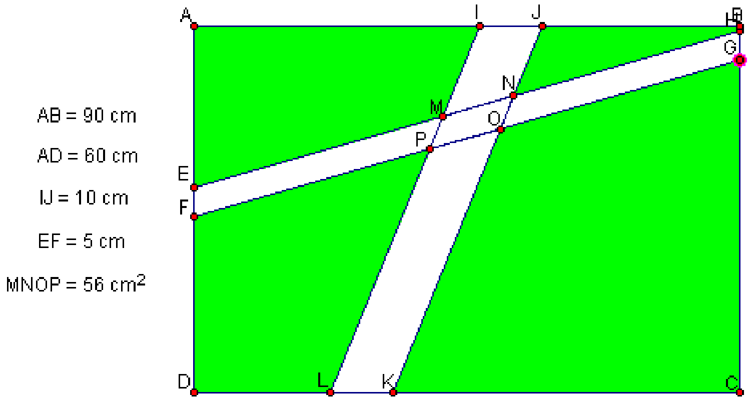


Fig. 9 The Area of MNOP in Condition1

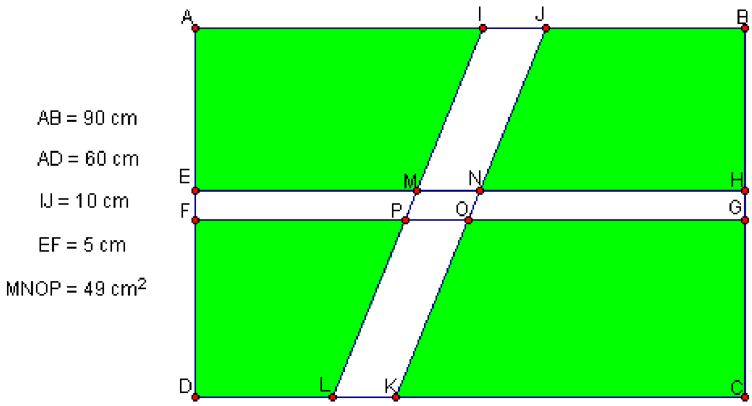


Fig. 10 The Area of MNOP in Condition2

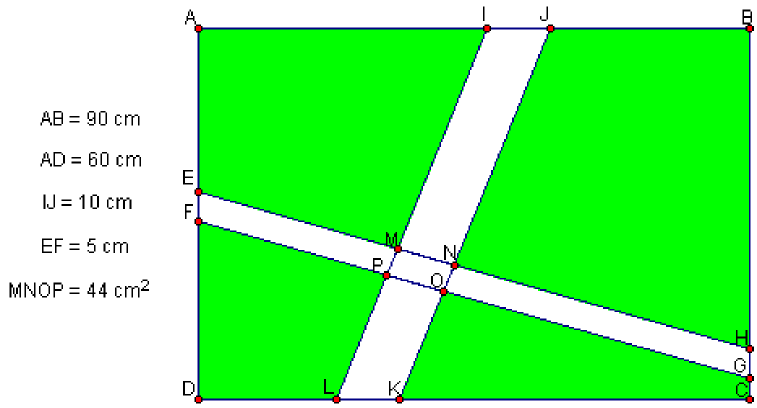


Fig. 11 The Area of MNOP in Condition3

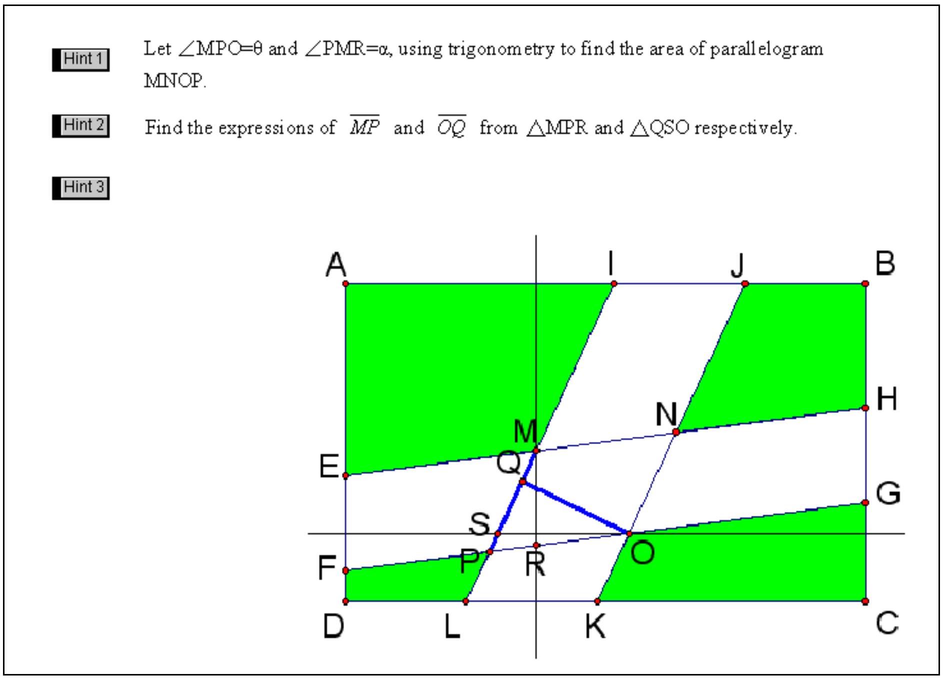


Fig. 12 The Powerful Three Hints

In this final stage, the representatives of each group will report back to the whole class. Each group got the new conjecture of the contradictory phenomenon and everyone concerned not only the fact of this phenomenon but also the reason why this phenomenon occurred. The teacher needed to listen carefully to the reports of the members who represent their groups and discuss the key points of their solutions. After all representatives had finished their reporting, the teacher needed to summarize different approaches to the pasture problem, eventually leading to a final. All eight groups in the class pressed the hint buttons to receive the support of computer multimedia. However, only two of the eight groups could build a model to solve this problem and explain the results of the contradictions and surprise. The other six groups stopped their progress at hint 2 or hint 3. From their report, it is observed that they can not reach to the final solution due to the belief of self-distrust and lack of time. The following is the solution provided by one of the two successful groups in explaining the strange phenomenon.

We define that the width of the vertical road is x (i.e. $\overline{IJ}=x$), and the width of the horizontal road is y (i.e. $\overline{EF}=y$). It is supposed that the included angle of the two roads is θ (i.e. ∠MPO=θ), and the included angle of the vertical road IJKL and $\overline{AD}$ is α (i.e. ∠PMR=α). We also construct that $\overline{OQ}$ is perpendicular to $\overline{MP}$, $\overline{MR}$ is parallel to $\overline{AD}$, and $\overline{OS}$ is parallel to $\overline{AB}$ (See Fig. 13). Observing △MPR, we can find that $\frac{y}{\sin\theta} = \frac{\overline{MP}}{\sin(180^\circ - \alpha - \theta)} \Rightarrow \overline{MP} = \frac{y \times \sin(\alpha + \theta)}{\sin\theta}$. Similarly observing △QSO, we can get that $\frac{x}{\sin 90^\circ} = \frac{\overline{OQ}}{\sin(90^\circ - \alpha)} \Rightarrow \overline{OQ} = x \times \cos\alpha$. Hence the area of the parallelogram MNOP is equal to

$$\begin{aligned} & x \times y \times \cos\alpha \times \frac{\sin(\alpha + \theta)}{\sin\theta} \\ &= x \times y \times \cos\alpha \times \frac{\sin\alpha \cos\theta + \cos\alpha \sin\theta}{\sin\theta} \\ &= x \times y \times \cos\alpha \times (\sin\alpha \cot\theta + \cos\alpha) \end{aligned}$$

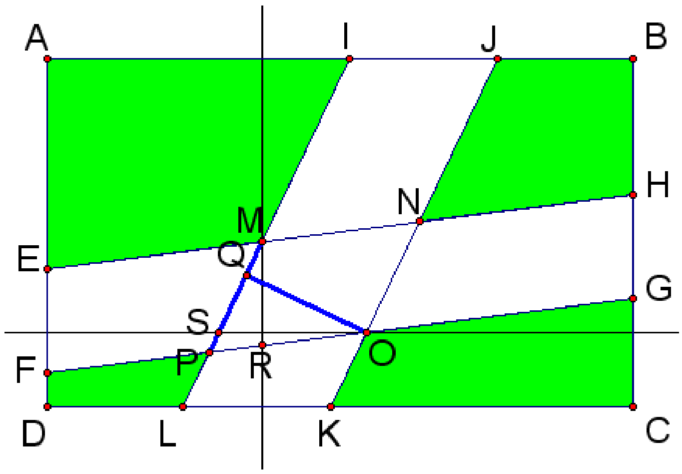


Fig. 13 The draft of the formal proof

From this formula, we can consider the following conditions of *the pasture problem*.

(1) When the included angle α of vertical road and $\overline{AD}$ is fixed, we can find that the larger the included angle θ is, the smaller the area of parallelogram MNOP becomes. This is because when θ increases, $\cot\theta$ decreases, where 0°<θ<180°.

(2) When the vertical road EFGH is parallel to $\overline{AD}$ (i.e. α=0°), the area of parallelogram MNOP is equal to $\overline{IJ} \times \overline{EF}$ (i.e. x × y).

(3) When the horizontal road IJKL is parallel to $\overline{AB}$ (i.e.$\alpha + \theta = 90^\circ$), the area of parallelogram MNOP is equal to $\overline{IJ} \times \overline{EF}$ (i.e. x × y) on account of $\cos\alpha \times \frac{\sin(\alpha + \theta)}{\sin\theta} = \cos(90^\circ - \theta) \times \frac{\sin 90^\circ}{\sin\theta} = \sin\theta \times \frac{1}{\sin\theta} = 1$

After finishing this teaching program, four students were interviewed; two of them were in the successful groups and the other two didn't reach to the final solution. All the interviewees claimed that they liked to have a similar curriculum in the future. Although *the pasture problem* was not easy to solve, they still felt interested in finding the solution due to the differences between the original guess and the new conjecture. They also changed the preconception that problem solving would take a few minutes. In fact, from the conjecture to a proof, they had to plan, execute, and monitor the solution through group discussion and guided discovery. This took them a long time. The two interviewees in the successful groups showed that they didn't know how to get the area of Parallelogram MNOP in the beginning. After getting hint 1, they really understood how to move on and started their job. Hint 2 and Hint 3 gave them more confidence in constructing the final solution through group discussion. This was because they could use these hints to monitor the progress of their final reports. The other two interviewees clarified that their efforts were stopped at Hint 2 or Hint 3, but they really understood more clearly about the nature of *the pasture problem* through engaging in the problem solving process. Especially when listening to the final reports of all groups, they shared their experience with others and learned more from the other groups.

4. Conclusions

In the context of our task, proof assumed the multiple roles of a verification of the truth of conjectures, an understanding of geometric relationships and an explanation, that is, giving insight into why this strange phenomenon works in the observed way. Technology support here plays an important role resulting in dissolving cognitive conflicts of mathematical proof. When the teacher poses *the pasture problem* with computer multimedia, students can guess and predict what the answer is and form original conjectures. After that students explore the problem with the virtual manipulative through the teacher's guidance. Technology makes them discover some new conjectures more easily, which do not coincide with their original conjectures. Because the original expectations are not fulfilled, cognitive conflicts occur and students try to find a solution to explain the new findings explored by them with technology support. Then the need to prove appears naturally. Students are eager to not only know the new conjecture is true but also understand why it is true. Next they use technology to check their reasoning and have more confidence in their new conjectures. Once their curiosity has been aroused and their judgments are supported through the computer support, students approach pencil-and paper proofs. But they still doubt that their reasoning is true. Students could check their new conjecture through the visual mode with computer support. Through group discussion and argumentation, they have to use deductive reasoning to construct a reason to explain the phenomenon. During this period, technology provides three powerful hints to help students model this problem and find out a proof to support the new conjecture. Therefore computer multimedia is a catalyst to help students combine their new conjectures with original conjectures and therefore lead to cognitive conflicts. During this teaching process, we could uncover that technology can support the expression of abstract mathematical ideas, build a context, learn by doing, modeling and scaffolding. The role of technology support and cognitive conflict in the problem-solving process is illustrated in Figure 13.

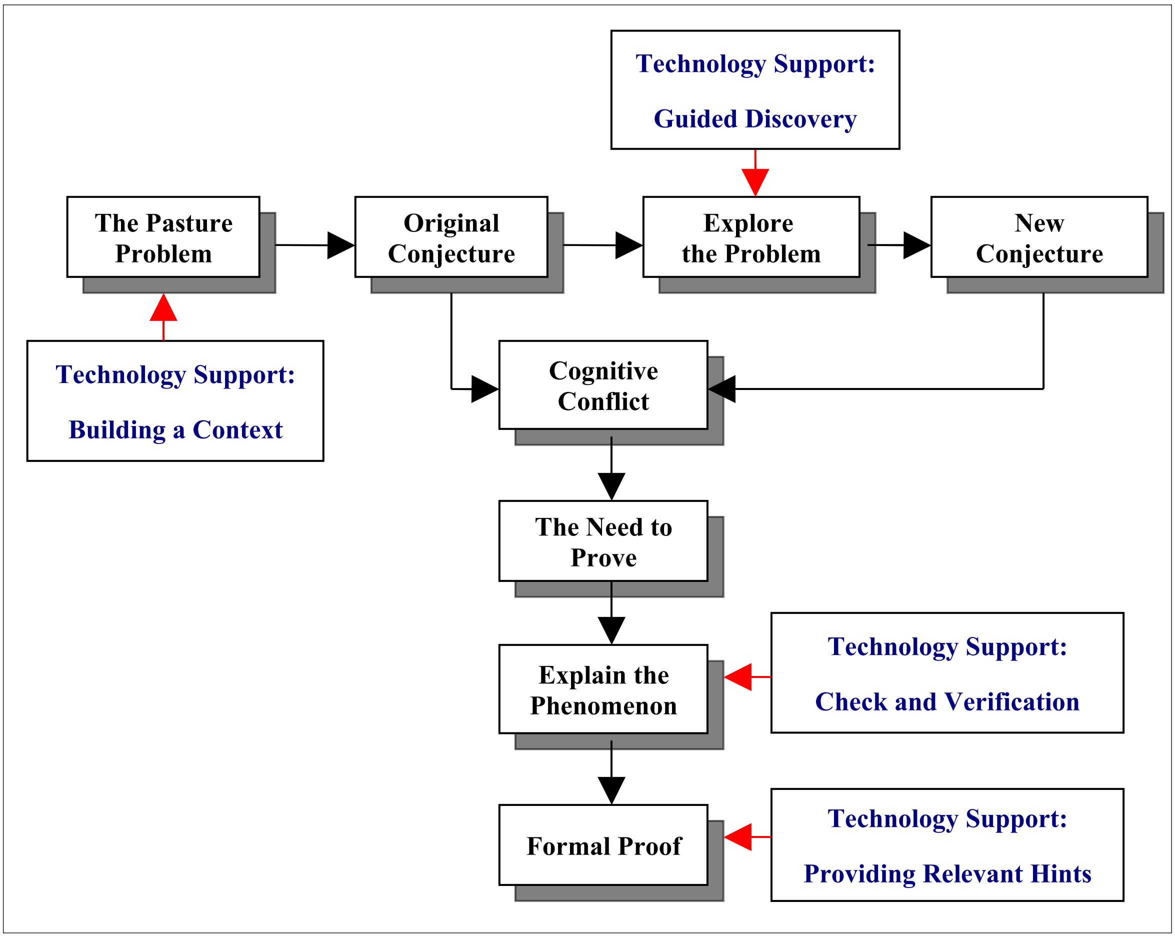


Fig. 13 The Role of Technology Support and Cognitive Conflict in the Problem-Solving Process

References

[1] Hadas, N., Hershlowitz, R., & Schwarz, B. B.(2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments, *Educational studies in Mathematics, 44*, 127-150.

[2] Clements, D. H., & Battista, M. T.(1992). Geometry and spatial reasoning. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). New York: Mac- Millan.

[3] Chazan, D. (1993). Instructional implications of students' understandings of the differences between empirical verification and mathematical proof. In J. Schwartz, M. Yerushalmy & B. Wilson (Eds.), *The Geometric Supposer: What is it a Case of?* (pp. 107-116). Hillsadle, NJ: Lawrence Eribaum Associates.

[4] Balacheff, N. (1988). *A study of students' proving processes at the junior high school level*. Paper presented at the 66th Annual Meeting of the National Council of Teachers of Mathematics, U.S.A.

[5] Holton, D., Oldnow, A., Porkness, R. & Stripp C. (2004). Investigations, proofs and reports. *Teaching mathematics and its applications, 23* (2), 97-105.

[6] Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior, 24*, 351-360.

[7] Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational studies in Mathematics, 48* (1), 2001, pp 101-119.

[8] Maher, C. (2002). How students structure their own investigations and educate us: What we've learned from a fourteen-year case study. In A. D. Cockbum & E. Nardi (Eds.), *Proceedings of the 26th Annual Meeting of*

the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 31-46). Norwich, England.
[9] Polya, G. (1945). How to solve it. Princeton: Princeton University Press.
[10]Schoenfeld, A. (1985). Mathematical problem solving. Orlando: Academic Press.
[11]Koedinger, K. R., & Anderson, J. R. (1993). Reifying implicit planning in geometry: Guidelines for model-based intelligent tutoring system design. In S. Derry & S. Lajoie (Eds.), *Computers as cognitive tools*. Hilladale, NJ: Lawrence Erlbaum Associates.
[12]Movshovitz Hadar, N. (1988). Stimulating presentations of theorems followed by responsive proofs. *For the Learning of Mathematics, 8* (2), 12-19.
[13]Dreyfus, T., & Hadas, N. (1996). Proof as answer to the question why. *Zentralblatt fiir Didaktik der Matematik, 28*(1), 1-5.
[14]Goldenberg, E. P., Cuoco, A. A., & Mark, J. (1998). A role for geometry in general education? In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 3-44). Hilladale, NJ: Lawrence Eribaum Associates.
[15]Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational studies in Mathematics, 44*, 55-85.
[16]Marrades, R. & Gutierrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*, 87-125.
[17]Mariotti, M. L. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational studies in Mathematics, 44*, 35-53.
[18]Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. J. Bishop, S. Mellin-Olsen & J. Van Dormolen (Eds.), *Mathematical Knowledge: Its Growth Through Teaching* (pp. 175-192). Dordrecht, Netherlands: Kluwer Academic Publishers.
[19]Garofalo, J., & Lester, F. (1985). Metacogintion, cognitive monitoring and mathematical performance. *Journal for Research in Mathematics Education, 16*, 163-175.
[20]Pugalee, D. K. (2004). A comparison of verbal and written descriptions of students' problem solving processes. *Educational studies in Mathematics, 55*, 27-47.
[21] Clark, R. C., & Mayer, R. E. (2002). e-Learning and the science of instruction: Proven guidelines for consumers and designers of multimedia learning. San Francisco, CA: John Wiley & Sons, Inc.
[22]Yuan, Y., & Lee, C. Y. (2004, July). Designing instructional tools by Flash MX to teach basic geometry concepts. In *Proceedings of the TIME-2004 Symposium*. Montreal, CA.