

# Technology-Based Manipulatives

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**Abstract:** One of the challenges confronting a Philippine mathematics teacher is the unavailability/lack of technology for use in the classroom. In this paper, we present technology-based manipulatives that provide opportunities for experimentation and discovery in the mathematics classroom as alternatives to the absence of technology. Models on Euclidean geometry, analytic geometry, trigonometry, and calculus will be discussed that can be used by students to observe and make conjectures on formulas and theorems.

## 1. Introduction

It is widely recognized that technology plays a major role in the teaching and learning of mathematics. Unfortunately, many public schools in the Philippines do not have access to these technology. Further, many teachers do not have experience in using computers and graphics calculators. This paper then provide ways by which teachers can innovate on readily available materials to simulate at least some of the benefits of using technology. Various technologies such as graphics calculators (GC), computer algebra systems (CAS), and dynamic geometry (DG) software each have an impact on how mathematics can be taught [5]. However, this paper shall focus on DG. According to the van Hiele Theory, the main reason traditional geometry curriculum fails is that it is presented at a higher level than those at which students are operating. The Van-Hiele Theory distinguishes between five levels of thought; the use of technology-based manipulatives is for example significant, in the levels of *ordering* (students understand interrelationships of figures) and *deduction*, (students begin to understand the significant of deduction, role of axioms, theorems and proof) [10]. The use of manipulatives based on DG hopes to benefit students in understanding mathematics to a larger extent. It has been reported that DGs provides positive impact not only on students' learning of geometric concepts [8] but on their mathematical reasoning as well [9]. In fact, throughout the activities mentioned, some possible open-ended questions are suggested to allow students to develop reasoning abilities with the aid of the visual aids. It is the aim of this paper to provide the same benefits to students, even with the absence of technology for use in the classroom. Further, DG lends itself naturally to the creation of technology-based manipulatives or instruments. These activities and manipulatives may not be new, but with the advent of DG software, it is also wise to go revisit these concrete and dynamic teaching aids.

This paper supports the *theory of active construction of knowledge* as attributed to Jean Piaget [11], where learners are actively engaged in building concepts. Students discover and gain

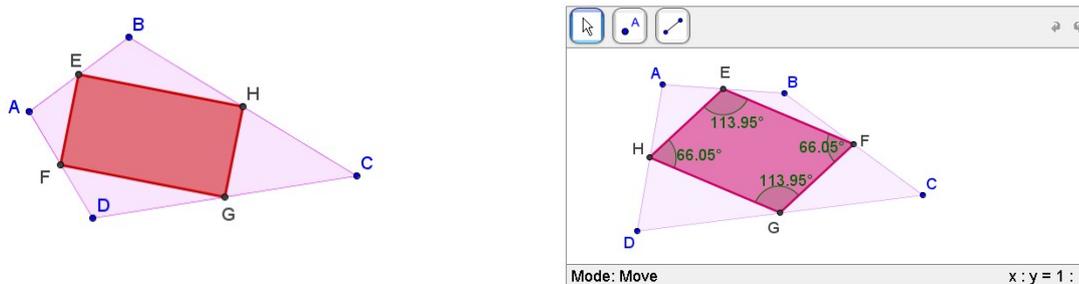
understanding of mathematical concepts by exploration and investigation, using technology-based manipulatives. In this paper, we also try to show that through technology, activities may be created that promote the use of multiple representations in acquiring mathematical knowledge, as suggested in [4]. This is done on the cognitive level—where for instance, graphs, geometric figures and diagrams represent abstract concepts. In the absence of technology, we model the ideas using indigenous materials that are available in the community.

To enable this work to be even more applicable to Filipino teachers, each electronic construction was done using Geogebra [15] and GRAPES [16], which are open-source dynamic geometry software. To access the files created using Geogebra, another free software Java [14], version 1.4.2 or later, should be installed in the computer. The HTML files can be viewed using Internet Explorer. To access the files created using GRAPES, the reader should first download the executable GRAPES file from [16].

## 2. Examples in Geometry

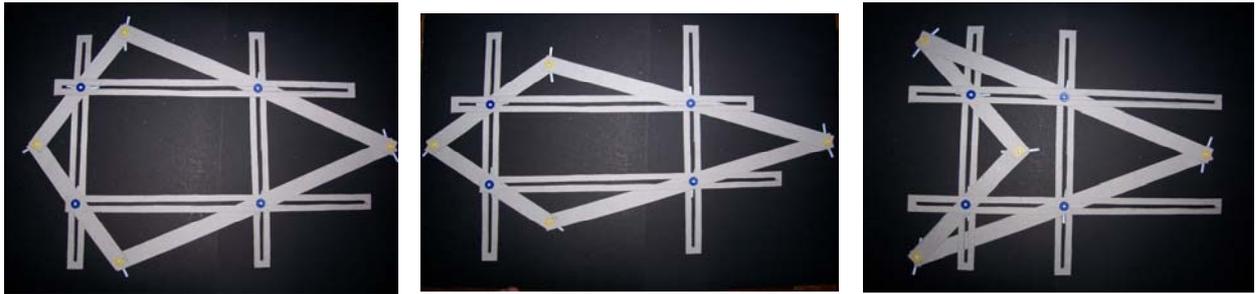
### A. Discovering Kite Properties

An investigation involving the midpoint quadrilateral of kites is found in [3]. Further, explorations on the midpoint quadrilateral of other types of quadrilaterals were performed using dynamic geometry software. In these activities, students were led to discover properties of the midpoint quadrilateral. Some expected student conclusions are that the midpoint quadrilateral of a quadrilateral is always a parallelogram, and that the midpoint quadrilateral of a kite is always a rectangle. Students were further challenged to find sufficient conditions on a quadrilateral so that its midpoint quadrilateral is always a rectangle. These activities were done using commercial DG software. Here, we do the same investigations using Geogebra. Open *kite.html* [18] and *kite2.html* [19].



**Figure 1:** Midpoint Quadrilaterals (Plots produced with Geogebra, [18] and [19])

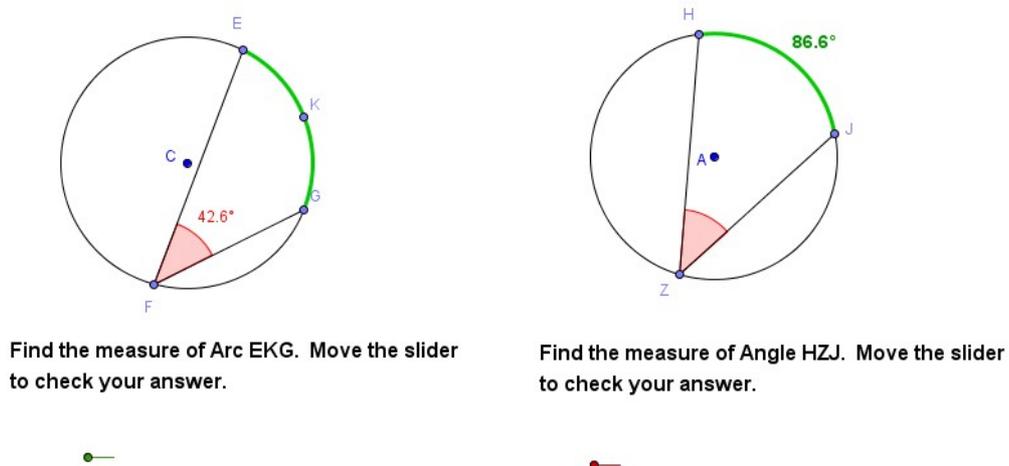
Alternatively, a simple manipulative using ordinary cardboard strips and pins can be used to demonstrate the same concepts. The idea of using strips to illustrate mathematics concepts can be seen in [1]. View the following movie files: *kite.mov* [20] and *kite2.mov* [21]. Some still shots of the manipulatives are shown below.



**Figure 2:** Illustrating the Midpoint Quadrilateral of a Kite Using Cardboard

### B. Angle Measurements in a Circle

It is a well-known theorem that angles which intercept the same arc are congruent, and their measures are half the measure of the intercepted arc. This is illustrated using a Geogebra construction. Further, these files can also be used to demonstrate the Theorem of Thales: that an angle inscribed in a semicircle is a right angle. Open [inscribed\\_angle\\_worksheet.html](#) [22] and [inscribed\\_angle\\_practice.html](#) [23].



**Figure 3:** Angle Inscribed in an Arc, (Plots produced with Geogebra [23])

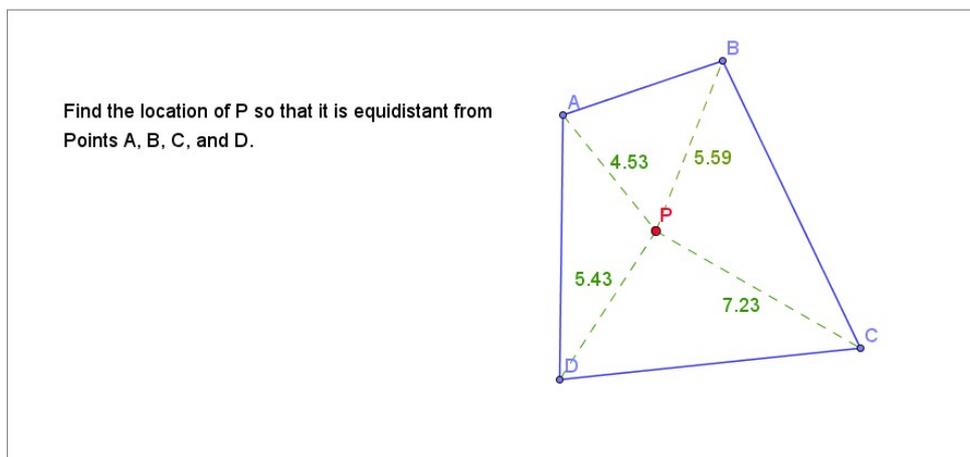
In the same way, an ordinary cardboard can be used to show that angles which intercept the same arc are congruent. View the following movie file: [semicircle.mov](#) [24]. The still shots are shown in Figure 4. Further, by attaching two more strips with endpoints at the center, one can also investigate the measure of the inscribed angle in relation to the intercepted arc.



**Figure 4:** Angles intercepting the same arc are congruent.

### C. Loci

DG Software could easily show a locus of points. For instance, the points equidistant from two points  $A$  and  $B$  are on the perpendicular bisector of the segment  $AB$ . Take the example from [6] of finding a point equidistant from four towns. Students are first asked to determine a point, if any, which is equidistant from four given towns, represented in Figure 5 by  $A$ ,  $B$ ,  $C$ , and  $D$ . Judging from experience, students will most likely perform this task by using the drag tool to drag point  $P$  so that it is equidistant from the four towns. Their approach is essentially by trial and error. However, by proper questioning, students can be led to discover that  $P$  must lie on the perpendicular bisector of each of segments  $AB$ ,  $BC$ ,  $CD$ , and  $DA$ . This exploration can be done on commercial DG software, but it could also be performed using Geogebra. See *Water1.html* [25]. Indeed, students were also able to see that the four towns lie on a circle, and this has led to interesting discussions. From an initial brute force method, this free software package allowed students to develop reasoning, as envisioned by Jiang [9].



**Figure 5:** Finding a point equidistant from four given points (Plot produced using Geogebra, [25])

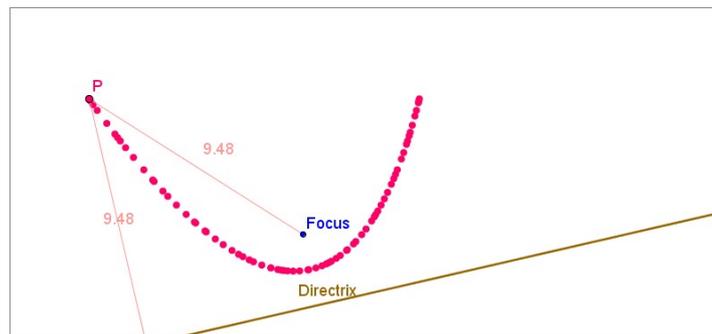
Teaching the locus of points can also be done creatively without using a computer. In [7], a treasure hunt activity was done to help students learn this topic. One given example is as follows. Points  $A$ ,  $B$ , and  $C$  are fixed landmarks in the school yard that could represent points (like a flagpole, tree, or trashcan). Small nails, representing treasures, are hidden below the ground.

Students are then given clues to find the nails. Sample clues are “One treasure is 15 feet from  $A$  and 22 feet from  $C$ ,” or “One treasure is equidistant from  $B$  and  $C$ .” Sometimes, as in both examples above, more than one point could satisfy the given clues and students have to check these points for the hidden treasure. This activity could be used as a motivational activity to introduce locus of points. The following activity is a modification of this Treasure Hunt, and its main problem is similar to the problem on the four towns given above. The teacher initially selects four landmarks, making sure that they all lie on a circle. Students could use tape measures to determine the point equidistant from these four landmarks. From here, students would again most likely resort to trial and error, but this activity motivates further discussion on perpendicular bisectors, thereby achieving a benefit from DG. This activity is well-received by students who enjoy doing activities outside the classroom.

### 3. Examples in Analytic Geometry

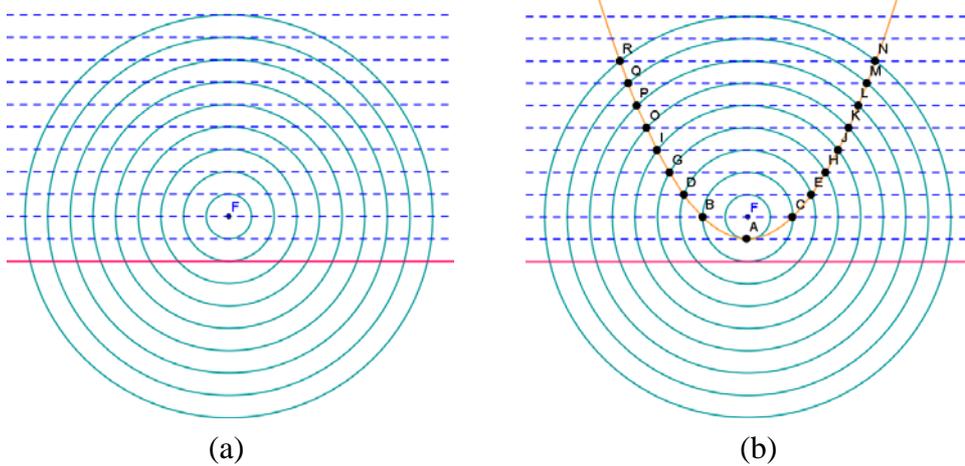
#### A. Parabola

The parabola is defined to be the set of all points equidistant from a fixed point, called the focus, and a fixed line, called the directrix. A Geogebra file can facilitate visualization and analysis of the locus of a parabola. See the following files: *parabola1.html* [26], *parabola2.html* [27], and *parabola3.html* [28].



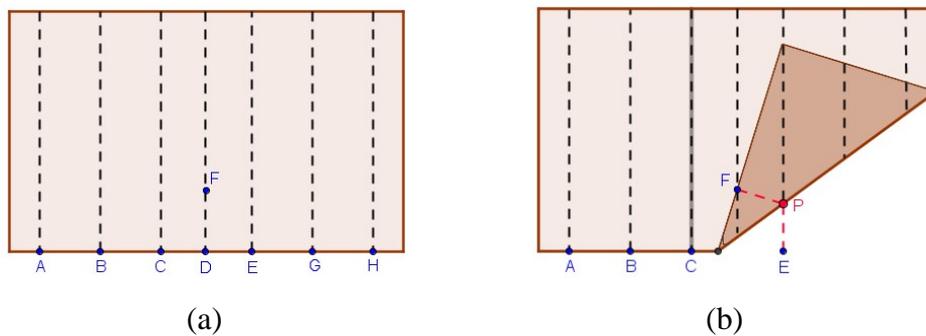
**Figure 6:** Screenshot of a Parabola (Plot produced with Geogebra [27])

One can also illustrate this definition even when a computer is unavailable. One can use concentric circles and parallel lines drawn on paper, as shown in Figure 7a. The red line is the directrix, and the focus  $F$  is at the common center of each circle. It is then easy to construct points equidistant from  $F$  and the directrix, as seen in Figure 7b. This idea was taken from [12].



**Figure 7:** Using concentric circles to investigate the parabola (Image Produced with Geogebra)

Another way to demonstrate the definition of the parabola even without a computer is by paper-folding. Indeed, many concepts in mathematics can be illustrated by folding paper. As in Figure 8a, one can start with a piece of paper with parallel segments, and a designated point  $F$  to represent the focus. A sheet of pad paper would work well for this activity. The base of the paper can be considered to be the directrix. Fold the paper so that the bottom of each parallel segment ( $A$ ,  $B$ , etc.) meets  $F$ . One such fold is demonstrated in Figure 8b. The intersection  $P$  between the dotted line and the crease made would be equidistant from the base of the paper and the focus. Hence, it is a point on the parabola.



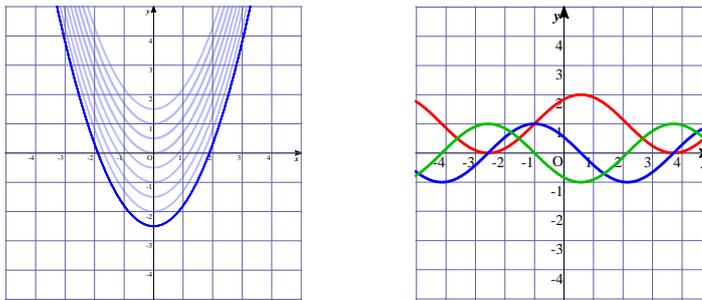
**Figure 8:** Using paper-folding to generate a parabola (Image Produced with Geogebra)

Much like the DG software, this paper-folding activity can allow students to visualize the definition of the parabola, which they might initially find abstract. Further, when students do this activity, they develop the sense that each point on the parabola satisfies certain conditions.

The reader interested in using concentric circles or paper-folding to investigate the ellipse and hyperbola could refer to [12].

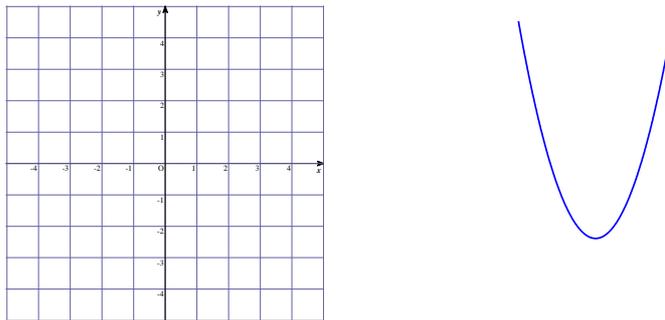
## B. Translations

One useful benefit of DG Software is that it allows the user to see effects of changing parameters in an expression. Because of this, one can use DG software to illustrate translations of graphs. For an example, see the GRAPES file *translations.gps* [29]. Another example would be an investigation on how changing the graph of a function changes the graph of its derivative. For reference, open the GRAPES file *derivatives.gps* [30]. Screen shots are shown in Figure 9.



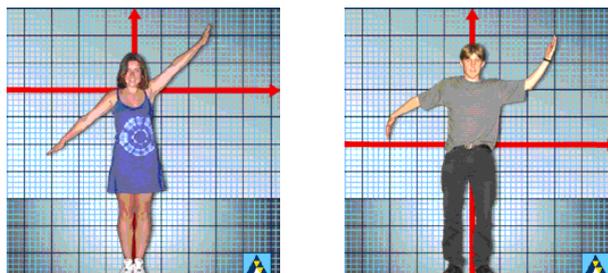
**Figure 9:** Translations and Derivatives (Plots produced with GRAPES [29], [30])

Unfortunately, only translations can be replicated if a computer is not available. One can use an overhead projector, and two transparency sheets to illustrate motion. One slide contains a fixed grid, while another slide contains a moving graph. See Figure 10.



**Figure 10:** A fixed grid on the left, and a graph which may be moved horizontally or vertically over the fixed grid to illustrate translations. (Plots produced with GRAPES)

One other idea is by using Aerobigraphs [13]. Here, students themselves illustrate transformations of graphs using bodily gestures. This is especially applicable to Filipino students, who are known to enjoy action-based activities, such as dancing. This method also has the advantage of being able to show graphical transformations other than translations.



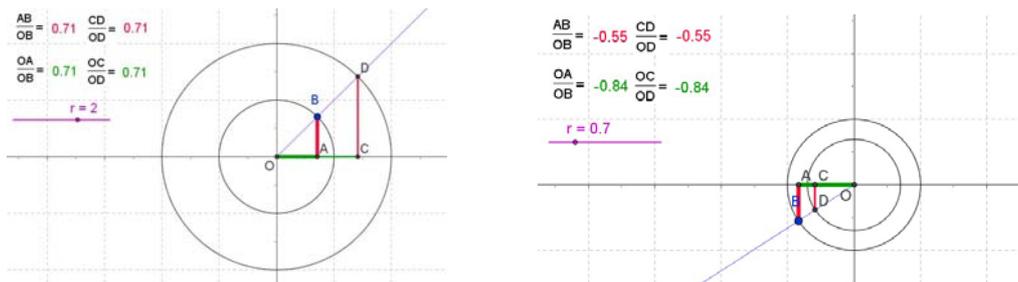
**Figure 11:** Using bodily gestures to demonstrate graphical transformations. (Image taken from [13])

## 4. Examples in Trigonometry

### A. The Unit Circle and the Sine and Cosine Ratios

In this section, some concepts in trigonometry are presented using DG Software. Toward the end, an alternative that does not need any computer is demonstrated.

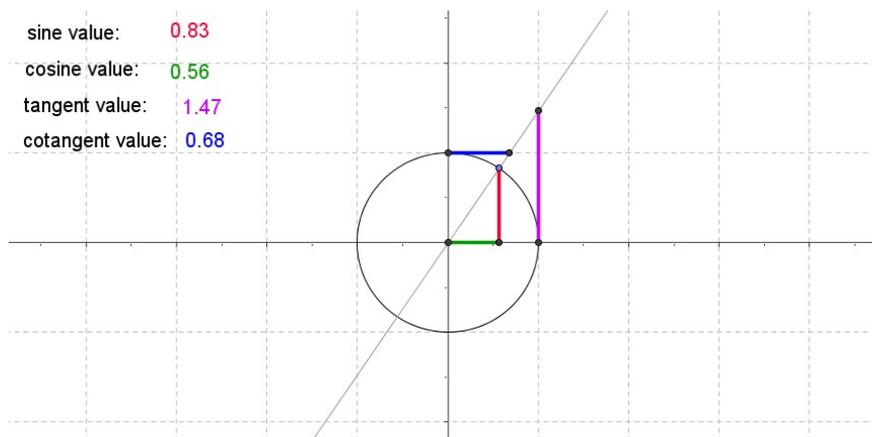
The unit circle is often used to illustrate the six trigonometric values. This convenient choice is based on the fact that the trigonometric ratios remain the same, even when circles with larger or smaller radii are used. Geogebra can illustrate this concept. See *UnitCircle.html* [31].



**Figure 12:** The Sine and Cosine Ratios Taken from Circles with Different Radii (Plot produced using Geogebra [31])

### B. A Geometric Interpretation of the Trigonometric Functions

As a second example, one may look at [14], which shows an applet that simultaneously shows the six trigonometric functions in relation to the unit circle. A similar applet, showing four of the trigonometric functions, was created using Geogebra. Open *FourTrigFunctions.html* [32].



**Figure 13:** Four Trigonometric Functions (Plot produced using Geogebra [32])

A similar idea is shown in [2]. This article was converted into a dynamic webpage using Geogebra. Open *SixTrigFunctions.html* [33].

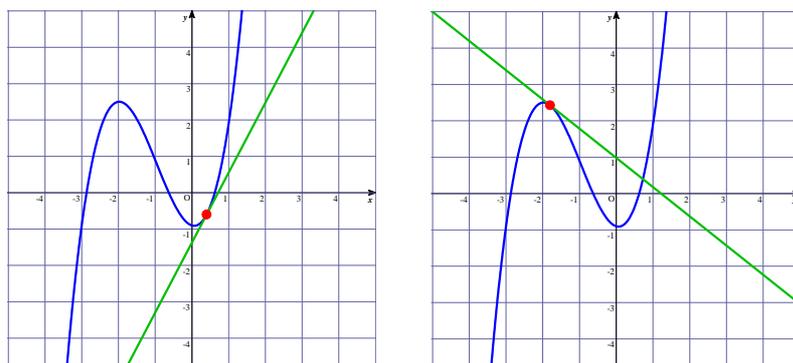


value undefined? What are the maximum and minimum values, if any, for each of the trigonometric functions? These questions can be answered, discussed, and visualized using this manipulative.

## 5. Examples in Calculus

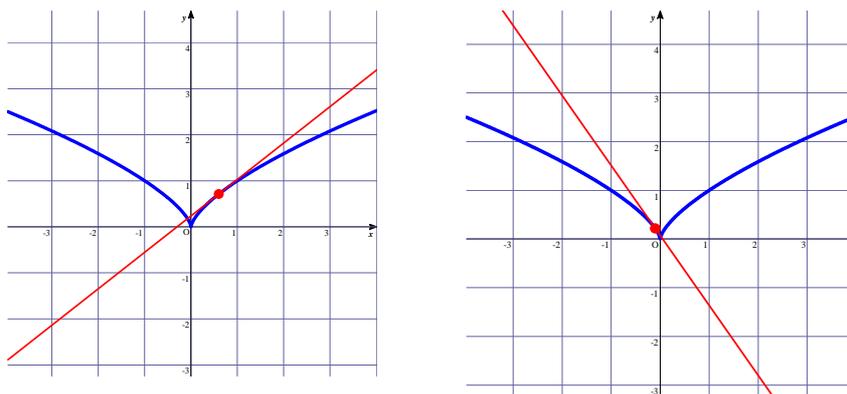
### A. The Derivative

The derivative is one important concept in Calculus. GRAPES can illustrate that the slope of the tangent line may vary as a point moves along the curve. Open *tangentlines.gps* [35].



**Figure 16:** Tangent Lines at Different Points of a Cubic Function (Plot produced using GRAPES [35]).

This software could also illustrate how a function can fail to have a derivative at a certain point as shown in *nondiff.gps* [36].



**Figure 17:** Non-differentiability at a Point (Plot produced using GRAPES [36]).

However, these effects can be shown even without DG software. One can use a transparency sheet and move a thin stick over it to show the tangent lines dynamically. Although this method has a disadvantage of not being able to show numerical slopes dynamically, students are still able to see clearly that its value *changes*. Again, students can still benefit from the usual questions afforded by DG software: At which point is the tangent line horizontal? Are there

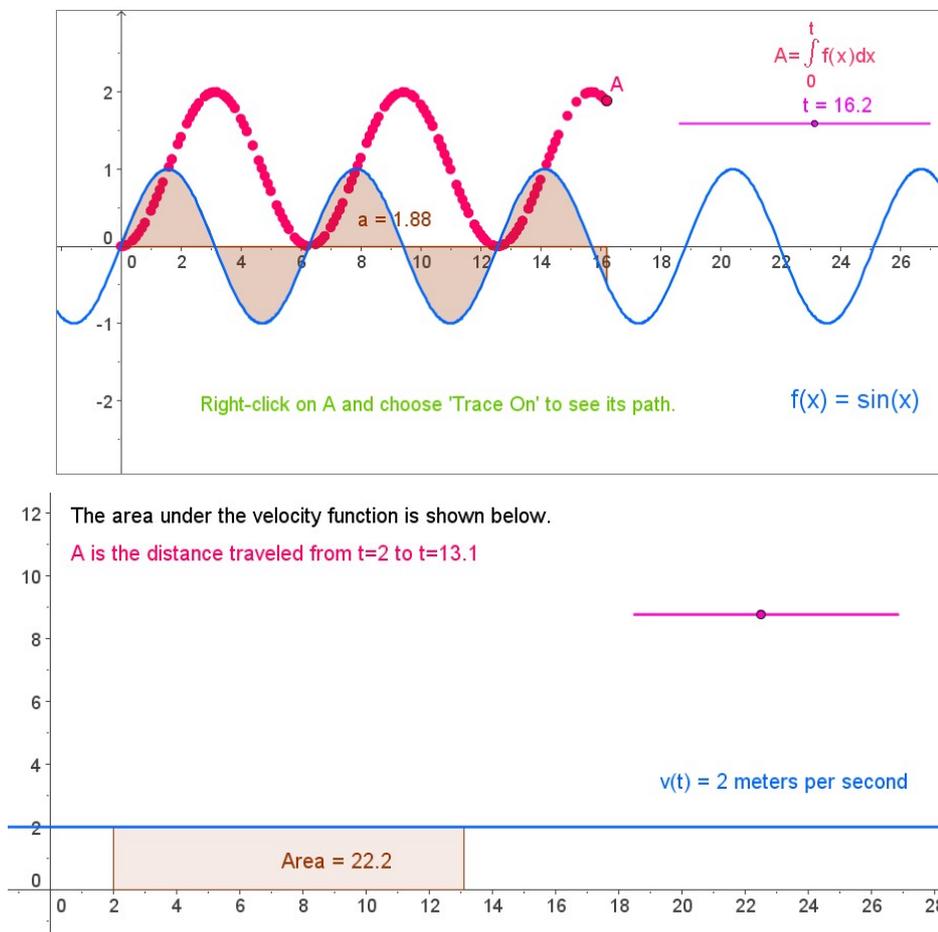
tangent lines that pass through the origin? What is the limit of the derivative from the left and from the right at the origin?

### B. The Fundamental Theorem of Calculus

The First Fundamental Theorem of Calculus is not easily grasped by students: From experience, it is very hard for students taking up calculus for the first time to understand the equation

$$\frac{d}{dx} \int_a^x f(t)dt = f(x).$$

Fortunately, DG software can show this theorem visually. Open the file *function1.html* [37]. By using the Previous and Next links in this *html* file, five other files may also be accessed, totaling six. Some screenshots are shown below.



**Figure 18:** The Fundamental Theorem of Calculus (Plot produced using Geogebra [37]).

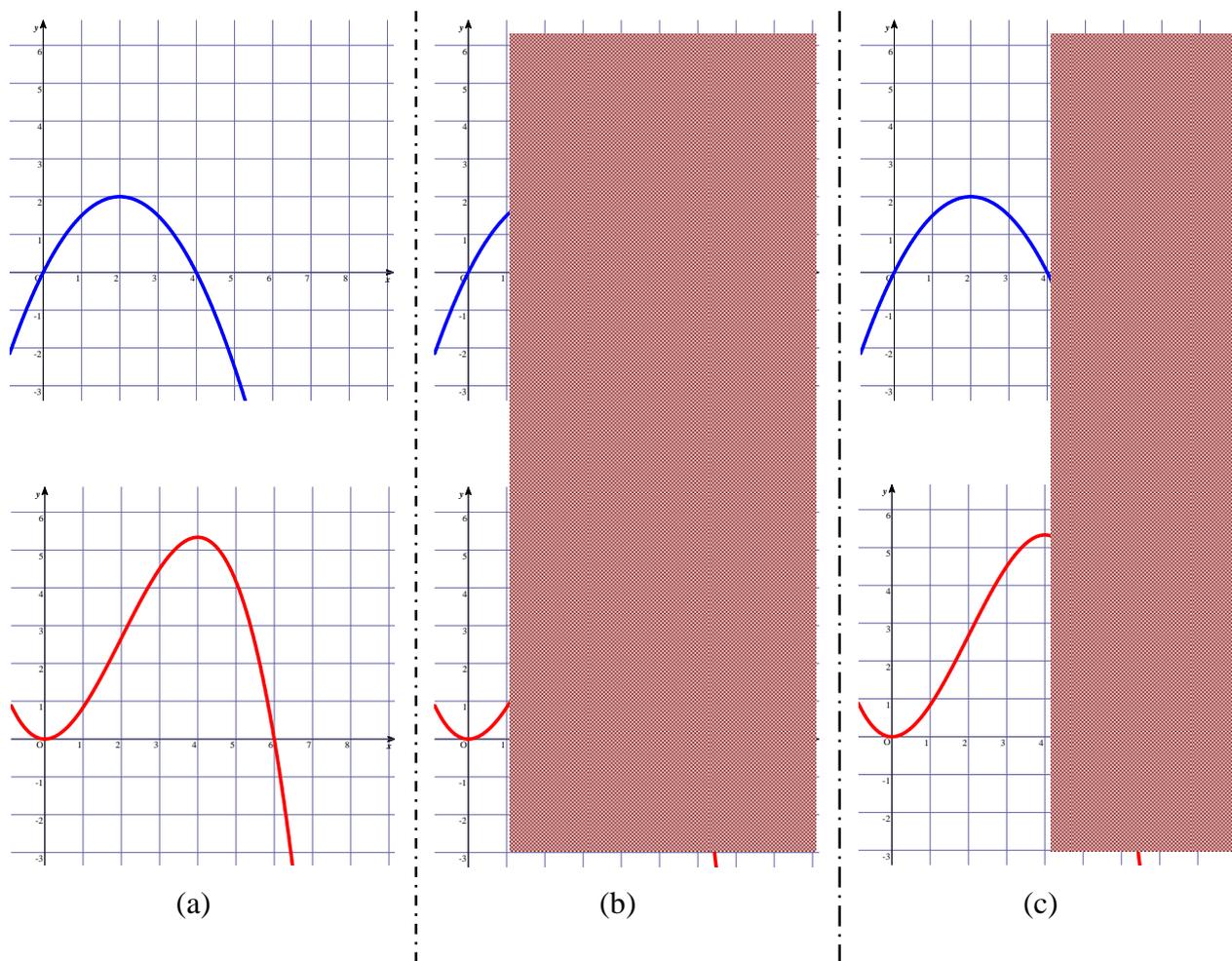
The first three applets illustrate the important idea behind the Fundamental Theorem of Calculus. Suppose that  $f$  is a continuous function of  $x$ , and one attempts to find the definite integral

of  $f$  over the interval  $[a, x]$ . Next, the values of this definite integral  $\int_a^x f(t)dt$  are taken as functions of  $x$  as well. This is a complicated idea and the use of DG helps students visualize this concept. By using the slider and the trace tool, students can see that the path formed by the values of  $\int_a^x f(t)dt$  consists precisely of values of the definite integral, and that this path turns out to be an antiderivative of the original function! These types of activities then allow students to see the important relationship between the definite integral and antiderivatives.

The last three applets meanwhile are examples of constructions that allow students to see that the area under the nonnegative velocity graphs given represents the distance traveled. A teaching suggestion would be to ask students to find the product of velocity and time during different time intervals, given the velocity curve. They should then be guided to discover that the products they computed can geometrically be represented by the rectangles in the construction. By doing so, students would see that in computing the displacement, one can use the strategy of finding the area under the velocity curve.

To illustrate this idea without using a computer, one may use transparencies again. The teacher prepares a transparency that contains the graph of a function  $f$ , and the graph of  $\int_a^x f(t)dt$  underneath it (refer to Figure 19a). Initially, these two graphs are covered using a long sheet of paper so that students cannot see them. Next, in Figure 19b, parts of the graph are slowly revealed to the students as the teacher slides the long sheet of paper to the right.. It must then be explained that the graph below represents the area under  $f$  starting at  $t = 0$ . The paper is continuously moved slowly to the right. Students could see that as long as  $f$  remains above the  $x$ -axis, then the graph of  $\int_a^x f(t)dt$  continues to increase. It can also be observed that the area starts to decrease once the graph of  $f$  falls below the  $x$ -axis. Finally, when the whole graph is revealed to students, it could be seen that  $f$  is the derivative of the area function shown underneath. This technique could be used to help students *discover* the theorem, rather than just *listen* to it being told to them.

These activities not only help students visualize concepts but these also help create in the students a positive disposition towards mathematics. Through these activities, students see mathematics as something that makes sense, and not just a set of formulas to read and memorize. For instance, students are able to sketch the graph of  $\int_a^x f(t)dt$  if they are given the graph of  $f$ . They are also able to observe that the derivative of the graph of  $\int_a^x f(t)dt$  (which they sketched) is the same as that of  $f$ . This is evidence that they understand the Fundamental Theorem.



**Figure 19:** The Fundamental Theorem of Calculus using a transparency (Image produced using GRAPES).

## 6. Conclusions and Future Work

It is widely recognized that the most schools in the Philippines do not have access to computers in the classrooms. However, this should not stop teachers from thinking of innovative ways to bring about the benefits of DG software to their students. Students may further be challenged to invent and construct their own manipulative and visual aids. This paper is not to be taken as a solution to all possible problems. Rather, it provides several examples so that teachers may use these as a springboard for more ideas of their own. Teachers might even utilize a mixture of both DG software and technology-based manipulatives, if these are available. It is hoped that future work would be done to come up with more teaching innovations that would benefit students the same way that DG can. The important thing is that teachers think of creative ways to use available resources, and to use these techniques to pose questions that would foster discovery and critical thinking. Indeed, one could use countless manipulatives and visual aids. However, if used solely to help students visualize, there is a danger that this would encourage even more passive learning. Thus, while teachers are encouraged to use these technology-based manipulatives, they are also urged to use these techniques as motivation for more meaningful class and group

discussions. It is further recommended that teachers use action research to study effectiveness of using these types of activities in their own classrooms.

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### Software Packages

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- [16] GRAPES, developed by Tomoda Katsuhisa, <http://www.criced.tsukuba.ac.jp/grapes/volume.html>
- [17] Java, a product of Sun Microsystems. <http://java.com/en/>

### Supplemental Electronic Material

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function6.html