Solutions to $8^{th}$ Order Polynomial Minimization Problem *

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Abstract
This paper presents a special canonical dual transformation method for finding global minimizer of a class of $8^{th}$ order polynomials. The method can be used to solve the $7^{th}$ order nonlinear algebraic equations that are their derivatives. Applications are illustrated by examples.

1 Primal Problem and its Canonical Dual

This paper presents a method to find global minimizer of the following $8^{th}$ order polynomial minimization problem

$$P_8(x) = \frac{1}{2}a_2(y_2(x))^2 + b_2y_2(x) + c_2 - dx,$$

where

$$y_2(x) = \frac{1}{2}a_1(y_1(x))^2 + b_1y_1(x) + c_1, \quad y_1(x) = \frac{1}{2}a_0x^2 + b_0x + c_0,$$

where $a_i, b_i, c_i (i = 0, 1, 2)$, and $d$ are given constants. In this paper, we assume that $a_i \neq 0 (i = 0, 1, 2)$. The criticality condition of $P_8(x)$ leads to a $7^{th}$ order nonlinear algebraic equation

$$(a_2y_2(x) + b_2)(a_1y_1(x) + b_1)(a_0x + b_0) - d = 0.$$

Clearly, direct methods for solving this $7^{th}$ order nonlinear algebraic equation are very difficult, or even impossible. Also, identifying the global minimizer of $P_8(x)$ is a fundamentally difficult task in global optimization. The canonical duality theory developed in [1] is a potentially useful methodology, and can be used to solve a relatively large class of nonlinear algebraic and differential equations. The associated triality theory can be used to identify both global minimizer and local extrema. In the recent paper [2], a special polynomial $P_n(x)$ with $b_i = 0 (i = 0, 1, 2, \ldots, n)$ has been studied. The purpose of this paper is to generalize this result to solve the $7^{th}$ order nonlinear equation and to identify global minimizer of $P_8(x)$.

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2 Canonical Dual Polynomial

From the pattern of the 8th order polynomial defined in (1), we can let

\[ y_2(x) = U_1(y_1(x)) = P_4(x) = \frac{1}{2} a_1(y_1(x))^2 + b_1 y_1(x) + c_1, \]  

(4)

which is a quadratic function of \( y_1 \). The dual variable \( s_1 \) of \( y_1 \) is defined as

\[ s_1 = U_1'(y_1) = a_1 y_1 + b_1. \]  

(5)

By solving this duality relation, we have

\[ y_1(s_1) = \frac{s_1 - b_1}{a_1}. \]  

(6)

Thus, by using the Legendre transformation

\[ U_1^*(s_1) = y_1(s_1) s_1 - U_1(y_1(s_1)), \]  

(7)

the Legendre conjugate function \( U_1^*(s_1) \) is well-defined as

\[
U_1^*(s_1) = -c_1 - \frac{b_1(s_1 - b_1)}{a_1} + \frac{s_1(s_1 - b_1)}{a_1} - \frac{(s_1 - b_1)^2}{2a_1} \\
= \frac{(s_1 - b_1)^2}{2a_1} - c_1.
\]  

(8)

In the same way, \( U_2 \) is set equal to

\[ U_2(y_2(x)) = \frac{1}{2} a_2(y_2(x))^2 + b_2 y_2(x) + c_2. \]  

(9)

The duality relation between \( y_2 \) and \( s_2 \) is

\[ s_2 = U_2'(y_2) = a_2 y_2 + b_2. \]  

(10)

Then, the Legendre conjugate of \( U_2(y_2) \) is

\[
U_2^*(s_2) = -c_2 - \frac{b_2(s_2 - b_2)}{a_2} + \frac{s_2(s_2 - b_2)}{a_2} - \frac{(s_2 - b_2)^2}{2a_2} \\
= \frac{(s_2 - b_2)^2}{2a_2} - c_2.
\]  

(11)

By the fact that

\[ U_2(y_2) + U_2^*(s_2) = y_2 s_2, \]  

(12)

the polynomial (1) can be written as

\[ P_8(x) = U_2(x) + c_2 - dx = y_2(x) s_2 - U_2^*(s_2) + c_2 - dx. \]  

(13)
Manipulation of equations (4) and (7) gives
\[ y_2 = U_1 = y_1 s_1 - U_1^*(s_1). \] (14)

Therefore,
\[ P_8(x) = s_2(y_1(x)s_1 - U_1^*) - U_2^* + c_2 - dx. \] (15)

Let \( \Xi(x, s_1, s_2) \) equal to equation (15). Further simplification yields
\[ \Xi(x, s_1, s_2) = s_2[(\frac{1}{2}a_0x^2 + b_0x + c_0)s_1 - U_1^*(s_1)] - U_2^*(s_2) + c_2 - dx. \] (16)

By solving \( \partial \Xi / \partial x \) we have
\[ x(s_1, s_2) = \frac{d}{a_0s_1s_2} - \frac{b_0}{a_0}. \] (17)

Substituting this \( x \) into (16) and setting the final equation equal to \( P^d(s_1, s_2) \), the canonical dual function can be obtained as
\[ P^d(s_1, s_2) = \Xi(x(s_1, s_2), s_1, s_2) = s_1s_2 \left[ c_0 - \frac{(s_1 - b_1)^2}{2a_0} \right] - s_2U_1^*(s_1) - U_2^*(s_2) + c_2. \] (18)

By combining (4), (6) and (10) and simplifying, the relation between \( s_1 \) and \( s_2 \) is given by
\[ s_2 = a_2 \left( \frac{1}{2}a_1 \left( \frac{s_1 - b_1}{a_1} \right)^2 + \frac{b_1 s_1 - b_1}{a_1} + c_1 \right) + b_2 = a_2 \left( \frac{s_1^2 - b_1^2}{2a_1} + c_1 \right) + b_2. \] (19)

By substituting (19) into (18) and letting \( s = s_1 \), the canonical dual function is finally obtained as \( P^d(s) = P^d(s, s_2(s)) \). By using Mathematica, \( P^d(s) \) can be calculated as
\[ P^d(s) = c_2 - \frac{a_2(s^2 + 2a_1c_1 - b_1^2)^2}{8a_1^2} + \left( b_2 + a_2c_1 + \frac{a_2(s^2 - b_1^2)}{2a_1} \right) \left[ c_1 - \frac{(s - b_1)^2}{2a_1} + s \left( c_0 - \frac{2a_0}{s(2a_1b_2 + a_2(s^2 + 2a_1c_1 - b_1^2))} \right)^2 \right] \] (20)

From the canonical duality theory [1], we have the following result:

**Theorem 1** If \( s \) is a critical point of \( P^d(s) \), then
\[ x(s) = \frac{d}{a_0s_2s} - \frac{b_0}{a_0}, \] (21)

with
\[ s_2 = a_2 \left( \frac{s_1^2 - b_1^2}{2a_1} + c_1 \right) + b_2 \] (22)

is a critical point of \( P(x) \) and \( P(x) = P^d(s) \). Moreover, if \( s \) is the largest critical point, then \( x(s) \) is a global minimizer of \( P(x) \).
3 Applications

3.1 Example 1

Let $a_0 = .1; a_1 = .51; a_2 = .5; b_0 = 0; b_1 = 0; b_2 = 0; c_0 = −6; c_1 = 0; c_2 = .5; d = 1.5$, we have

$$P_8(x) = 0.5 − 1.5x + 0.0162563(−6 + 0.05x^2)^4. \tag{23}$$

This is a nonconvex polynomial of degree eight with one global minimizer, one local minimizer and one local maximizer as shown in Fig. 1.

![Figure 1. Nonconvex $P_8(x)$ with one global minimizer, one local minimizer and one local maximizer.](image)

The canonical dual of this $P_8(x)$ has a very simple form

$$P^d(s) = 0.5 + 0.490196(−6 − \frac{46.818}{s^6})s^3 − 0.720877s^4. \tag{24}$$

Its graph is shown in Fig. 2.

![Figure 2. Canonical dual polynomial $P^d(s)$.](image)

The criticality condition $P^d(s) = 0$ has three real roots

$$s_1 = 1.32632 > s_2 = −1.5917 > s_3 = −3.02909. \tag{25}$$

From equation [21], we have $x_1 = 13.1154$, $x_2 = −7.58817$, and $x_3 = −1.10099$. It is easy to check that

$$P(x_1) = −18.4294 = P^d(s_1) < P(x_2) = 13.4246 = P^d(s_2) < P(x_3) = 22.3811 = P^d(s_3). \tag{26}$$
This shows that $x_1$ is a global minimizer, $x_2$ is a local minimizer, and $x_3$ is a local maximizer (see Fig. 3).

![Figure 3. Canonical dual polynomial $P_d(s)$.](image)

### 3.2 Example 2

In Example 1, if we let $a_0 = .1; a_1 = .51; a_2 = .5; b_0 = -.1; b_1 = -.2; b_2 = 0; c_0 = -7.5; c_1 = -5; c_2 = .5; d = -1.5$, the graph of the polynomial $P_8(x)$ is shown in Fig. 4.

![Figure 4. Nonconvex $P_8(x)$ with $c_1 = -7.5$.](image)

In this case, the canonical dual function $P_d(s)$ has a total of seven critical points (see Fig. 5).

![Figure 5. Canonical dual polynomial $P_d(s)$.](image)

$$s_7 = -4.03877 < s_6 = -2.58867 < s_5 = -1.82974 < s_4 = -0.537399 < s_3 = 0.467103 < s_2 = 2.04221 < s_1 = 2.43476.$$
By the Theorem 1 we know that $x_1 = -14.9475$ is a global minimizer of $P(x)$ and $P(x_1) = P^d(s_1) = -21.7721$. (see Fig. 6).

4 Conclusions

The results of this paper can be used to find the all critical points of the eighth polynomial derivative of equation (1). One advantage of using this way to find the critical values of $P_8$ is that, when the derivative of $P^d(s)$ is set to 0 and the critical values of $P^d(s)$ are found, the critical value with the largest value of $s$ is the absolute minimum.

Supplementary File

A Mathematica file is available for those who wish to experiment with these techniques.

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References
