Teaching and Learning Concrete and Theoretical Arithmetic through Technology

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Abstract
In this work we analyse the role of technology in the solution of theoretical and concrete arithmetic problems. In particular, we consider the role of ARI-LAB 2 system that makes available mediating tools (Microworlds) for the construction of solution processes for these two types of problems.
Microworlds of Ari-Lab-2 allow the student to manipulate interact with computational objects receiving various kinds of feedback. We present some results concerning the use of ARI-LAB-2 Microworlds for the development of competencies both in concrete and theoretical arithmetic and we discuss the pedagogical and educational potential of this system.

Introduction
For ancient Greeks of the classical age, Arithmetic was the theory of numbers, namely the study of numbers from a theoretical point of view. Instead of Arithmetic, ancient Greeks used the term "Logistica" to indicate exactly the art of computation and of solving concrete problems. Hence, Logistica deals with "numbered things" rather than "numbers".
Nowadays we use the term "concrete arithmetic" to indicate practical activities with numbers. We distinguish it from "theoretical arithmetic" which, instead, deals with abstract properties of numbers. This distinction is reflected also in the educational field where arithmetic teaching has been historically characterized by a separation of contents and competencies related to these two different uses of numbers. Thus, the nature of problems faced in concrete and theoretical arithmetic as well as their solution processes and their validation tools, are different.
Students' obstacles and difficulties in solving these problems are different. This work focuses on some research results concerning the use of ARI-LAB 2 to overcome them.

Concrete and theoretical arithmetic problems
The following problems are paradigmatic examples of the different nature that characterizes concrete and theoretical arithmetic.
<u>Task 1</u>
Greta has 4 Euro and 42 Cents in her purse. She wants to buy a book that costs 7 Euro and 75 Cents. How much more money does Greta need to buy that book?
<u>Task 2</u>
Consider the number corresponding to the result of the following numerical expression: 420+168+63. Show that three numbers of the expression are respectively divisible by 7 and 3. Moreover, show that 7 and 3 are divisors of the result of the expression too.

Task 1 belongs to concrete arithmetic and it evokes an every day situation. Chevallard has highlighted that historically the solution of a concrete problem is based on the use of verbal language and on the potential offered by a numerical system [4]. An example of solution by means of verbal language is the following :

"*From 4 Euro and 42 Cents to 4 Euro and 50 Cents there are 8 Cents. From 4 Euro and 50 Cents to 7 Euro and 50 Cents there are 3 Euro and to 7 Euro and 75 Cents there are 25 Cents yet. Hence to buy the book Greta needs 3 Euro and 33 Cents more.*

As we can see, this solution is a discourse that exploits properties of the European currency system. The concrete aspects of the situation play an important role in the construction of a solution for these problems. The control of its correctness is based on a practical validation with respect to the concrete situation or to a shared common sense that emerges through the reading of the discourse-solution. Some researchers have defined this type of solution as oral solution [4; 3]. After the advent of the algebra notation (in the XVI century) and the use of the operation signs also in the arithmetic domain, a solution for this problem can be expressed by a chain of arithmetical expressions involving numbers joined by the operation signs (i.e: 4.42+0.8+3+0.25=7.75 or 7.75-4.42=3.33). We note that numerical expressions make sense as solution of a concrete arithmetic problem situation in virtue of the fact that, in their structure, they reflect the way to exploit the potential of a numeric system presented by an oral solution.

In this framework, obstacles and difficulties met by students in the solution of concrete arithmetic problems concern:

- the construction of solution strategies exploiting the potentiality offered by a numerical system;
- the attribution of meanings to the signs of the arithmetic operations in the solution of additive and multiplicative problems and the ability to use them effectively to condense the strategy of the oral arithmetic.

Task 2 represents a theoretical arithmetic problem; it recalls an abstract situation where it is necessary to demonstrate a specific property (divisibility) of the integer numbers applied to a numerical situation expressed in a symbolic form. Historically, the solution of this kind of problems has been based on the use of verbal language or on the exploitation of specific representations (very often of geometric nature) to give a perceptive evidence of the specific abstract numerical property one wants to study[1]. In this type of problems the validation of the solution depends on both the method and the system of representation used.

Nowadays the solution of a task 2-like problem is represented by the construction of a numerical expression that interprets the theoretic arithmetic situation, and by means of a deductive chain of symbolic transformations aims at reflecting the structure of the property one wants to demonstrate in the structure of the formal expression.

In this framework, obstacles and difficulties met by students in the solution of theoretical arithmetic problems concern:

- the construction of an effective representation of the problem through the use of the symbolic arithmetic language;
- the use of the rules of symbolic transformations to demonstrate a specific property.

We observe that symbolic arithmetic language can be used to solve and validate both problems of concrete arithmetic and of theoretical arithmetic. In the former case symbolic language is used in a procedural way (to perform the computations that are necessary to find the result), while in the

[1] As an example, consider the Pythagorean arithmetic in which numbers were represented by stones disposed in geometrical figures. The arrangement of the stones can represent specific numerical properties, for example that two successive triangular numbers (triangular numbers: 1, 3, 6, 10, 15, …) form a quadratic number (quadratic numbers: 1, 4, 9, 16, 25…).

latter, the symbolic language is used in a relational way (to highlight the numeric property as reflected in the form of the expression). The development of a genuine arithmetic knowledge involves the capability to use the symbolic language with this double functionality.

The role of technology

To analyse the role of technology in developing concrete and theoretical arithmetic competencies we hypothesize that it is possible to exploit the characteristics of visualisation, interactivity, dynamicity and computation offered by technology:

- To transform a numeric system into a digital numeric Microworld that is characterized by new operative and representative potentialities and that is able to bring under the control of perception numerical procedures and concepts involved in the construction of resolution strategies for concrete arithmetical problems [2].
- To transform a system of symbolic manipulation (namely signs, rules, conventions and principles governing the symbolic transformation) into a digital symbolic manipulator Microworld that is characterized by new operative and representative potentialities and that is able to bring under the control of perception the hierarchical structure of symbolic expressions, the rules that can be applied to an expression or to a part of an expression preserving the equivalence and the effects produced by their application. [5].

Following these transformations operated by technology new forms of mediation for the cognitive development involved in the construction of competencies concerning the solution of concrete and theoretical arithmetic problems are made available.

We discuss our hypothesis through two Microworlds incorporated into ARI-LAB-2 system, namely the Euro Microworld and the Symbolic Manipulator Microworld.

The Microworlds of ARI-LAB 2 as new support for concrete and theoretical arithmetic

ARI-LAB 2 [2] is a system for the development of arithmetical capabilities (EC project Itales IST-2000- 26356). ARI-LAB-2 makes available different environments to support visualization, elaboration and communication of arithmetical knowledge in a socio-constructivist pedagogical perspective. The system is designed for a computer-equipped school laboratory in a local network.

In ARI-LAB 2, problem solving is carried out by exploiting the action, representation and communication possibilities made available by the different environments integrated in the system. They are used to develop the solution process (Microworlds), to describe and present problem solutions (Solution sheet), and to allow communication between users (Communication Environment).

ARI-LAB 2 makes available 10 Microworlds to develop solutions of problems in a classroom context. A Microworld models resources and constraints of concrete and theoretical arithmetical situations by means of interactive computational objects. Through this interaction students receive a feedback that can be interpreted as a mathematical phenomenon. Some Microworlds (Euro, Calendar, Spreadsheet, Graphs) were designed to shape concrete arithmetical context and situations, such as sale or time measurement. Other Microworlds (Fraction Microworld and Manipulator) model context and situations of theoretical arithmetic.

In the Solution Sheet it is possible to elaborate the solution process enacted within Microworlds and transform it into a product to reflect on and to share with others students.

During the solution process students and teachers use a communication environment on the local network to exchange messages, send and receive problem solutions as in a chat (for more information on ARI-LAB2 access: http://www.itd.cnr.it/arilab/).

Concrete arithmetic and the role of the Euro Microworld

Progress in cognitive development related to the construction of concrete arithmetic problem-solving strategies cannot be understood without considering the context they are developed in [6].

Studies concerning cognitive development in concrete arithmetic problem solving have demonstrated the importance of linking school mathematics to fields of experience [1]. The fields of experience are characterized by specific tools, knowledge, rules, values, principles, behavioural schemes that determine the cultural-specific symbolic forms and practices through which the potentialities of a numeric system can be easily exploited in the construction of the solution. For example, the field of "purchase and sales" (with the use of a currency numeric system based on coins and banknotes and of their monetary computation potential) or of a "measure of time" (with the use of a calendar numeric system and of its potential in counting intervals of days, weeks and months) are fields of experience that embed important aspects of concrete arithmetic based on the exploitation of numeric systems that are specific of these fields.

In this framework we think that technology can be used to transform the specific numeric system used in a field of experience into a new digital numeric Microworld that is characterized by new operative opportunities and new representative dimensions.

The new operative and representative opportunities of the digital numeric Microworlds might have a great importance for the educational perspective as they can suggest to students possible courses of action to be taken when solving the concrete problem at hand and/or to encourage the emergence of objectives to that end. Let us consider the Euro Microworld of Ari-Lab-2.

Euro Microworld models resources and constraints of "purchase and sale" field by means of computational objects which reify the coins and banknotes of the current European currency. The representative and operative possibilities of Euro Microworld are: to generate, every time, virtual coins in a working space choosing in the interface the more appropriate one to represent the problem situation, to move a coin or a group of coins in the working space in order to obtain an appropriate representation to solve the problem, to cancel generated coins.

While interacting with these computational objects, the user receives various kinds of feedback that may foster the emergence of goals for problem solution and the construction of meaning for the strategies developed. In particular, we stress the importance of two kinds of feedback that outside the Microworld can be available only interacting with the teacher or with a more capable peer:

- Changing a coin or a group of coins with others of the same value. The system provides a feedback ("too few", "too many") useful to orient and modify the solution strategy.
- Hearing the value corresponding to a selected coin or group of coins by means of a voice synthesizer.

We can observe that all these visual spatial and interactive features of the Euro Microworld can have a great importance for the cognitive development involved in the exploitation of the potentialities of a numeric system to solve a concrete arithmetic problem. Let us consider a solution to task 1 taken from experimentation with a grade 2 primary class.

To solve the problem a student generated the coins necessary to purchase the book and the coins in Greta's pursue (Figure 1) in the Euro Microworld working space and used a voice synthesizer to validate the constructed amounts.

Figure1: Coins necessary to purchase the book and coins in Greta's purse generated in the Euro Microworld working space.

Then the student seemed stumped and, for a certain period of time, appeared unable to find a solution strategy. At this point, two kinds of solution strategies are possible:

- the completion strategy, adding to Greta's coins the money needed to reach the cost of the book;
- the take away strategy, taking away Greta's coins from the cost of the book in order to find the money Greta needs to buy the book.

This latter strategy was chosen by the student who cancelled Greta's coins from the working space and used the monetary change function to change the amount corresponding to the cost of the book in order to separate the amount corresponding to Greta's coins from the amount Greta needs to buy the book. In our experimentation we have observed that several students do not respect the monetary equivalence when they use the change function. We note that students can change a coin to obtain the monetary value that they consider necessary for an immediate objective of solution. As a matter of fact, in our case, students can change 5 euros with 2 euros to add to the 2 euros coin belonging to the amount of the cost of the book and already represented in the working space.
We note that the achievement of 4 euros is considered by the students an immediate objective of solution.
The feedback provided by the change function ("too few" - Figure 2) obliged the student to respect the monetary equivalence and to add 3 euros more to concretely perform the monetary change.

Figure 2: Feedback provided by the monetary change function of Euro Microworld. Note that during this operation the working space colour changes and only the coins involved in the change stand out.

After this correction, the solution is brought to the end by using the possibility to move coins with the mouse in the working space, in order to separate 4 Euro and 42 Cents from the coins representing the cost of the book (Figure 3).
When the student physically separated the 4 Euro and 42 Cents from the total amount (emergent goal), they attributed a double meaning to the coins representing 4 Euro and 42 Cents: both as symbols that contributed to the cost of the book and as symbols that represented the amount possessed. It is important to note that the recognition of this double meaning is the result of appropriation of the coin’s cultural forms which, in the activity, is specialized as a symbolic vehicle to serve new particular cognitive functions [6].

Figure 3: The amount of 4 Euro and 42 Cents is separated from the coins representing the cost of the book by means of the possibility to move coins with the mouse in the working space.

The voice synthesizer can be a useful tool to validate the performed strategy verifying that the amount represented in the above part of the working space corresponds to Greta's coins amount (Figure 4); that the total amount represented in the working space corresponds to the cost of the book and hence that the amount represented in the bottom part of the working space is the solution of the problem. This validation process is based on the respect of the monetary equivalence.

Figure 4: Merlin wizard Avatar of the Voice synthesizer. He reads the total amount represented in the working space.

We observe that the voice synthesizer and the function of monetary change of the Microworld can support didactical practices in which students can investigate the notion of numeric equivalence that is crucial in the development of a solution strategy for concrete arithmetic problems. Feedback offered by the voice synthesizer and by the function of monetary change provides the support for the development of the concept of numeric equivalence in terms of monetary equivalence and guidance in the solution strategy. This is an example of use of the potential offered by a digital numeric system to solve the problem.

The operative and representative opportunities of this Microworld make it possible to express the solution process in terms of relationships and spatial-dynamic operations involving coins and groups of coins. This occurs through the possibility to group and to separate coins offered by the Microworld. This characterises the arithmetic that develops when working within the Microworld-modelled field of experience and distinguishes it from the arithmetic traditionally adopted in school to tackle problems of the same sort, since the very beginning, is based on formal expression involving numbers and operations.

The capability to construct solution strategies exploiting the potentialities of the digital numeric Microworld is crucial for the development of oral arithmetic skills, as described in a previous section of this paper. These skills can be developed over time through the conversion of solutions produced in the Euro Microworld into solutions based on verbal language. The conversion of the Microworld-based solution strategy into a written language-based discourse is fundamental for the interiorization of the action performed in the Microworld. Many meanings related to practices undertaken in Microworlds can be effectively brought to the student’s consciousness through conversion in another register of representation. For this reason teachers asked to verbalize the solution strategy [2].

Also the ability to convert the oral solution strategy into an arithmetic expression can be developed progressively. With the teacher’s help, students progressively learn to use arithmetic symbols to express the meaning of the strategies based on oral arithmetic and Microworlds. At the beginning arithmetic symbols can be introduced by the teacher as a mean to synthesise the reasoning based on oral arithmetic or to translate the actions performed in the Microworld.

Through this didactic practice mediated by the teacher, students gradually learn to master the meaning that arithmetic symbols assume in problem solving situations, reorganizing their Microworld-based solutions in accordance with the potential the new symbols offer.

Theoretical arithmetic and the role of the Symbolic Manipulator Microworld

In compulsory education the difficulty of passing from practical activities which refer to every day experiences to theoretical activities which refer to a set of principles and rules of mathematics is well-known.

Let us consider the previously presented task 2 which requires the analysis, the transformation and interpretation of the numerical expression 420+168+63 considered in the set of natural numbers. We note that the solution of this task by means of the Symbolic Manipulator Microworld forces students to refer constantly to theoretical aspects of symbolic transformation.

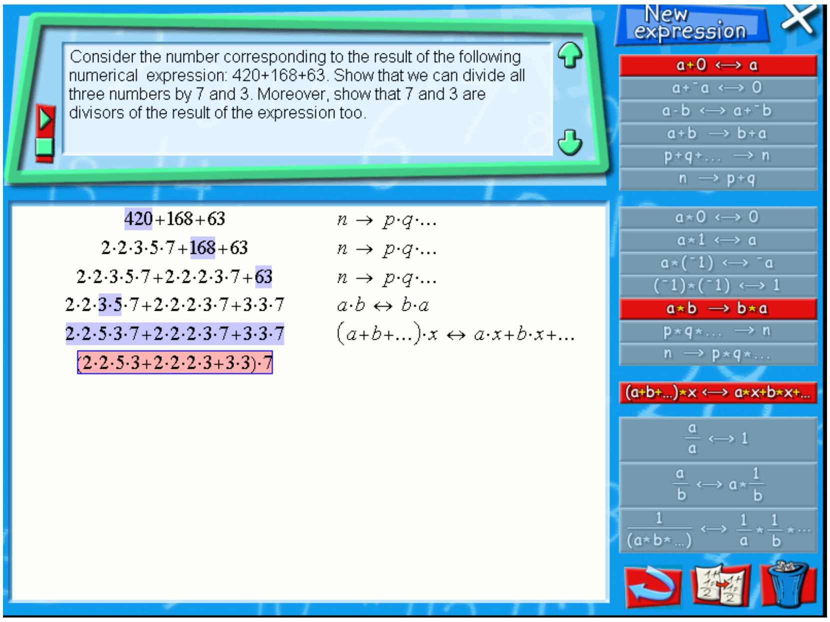

Figure 5: Transformation of the numerical expression in interface of Manipulation Microworld.

In Figure 5 the interface of the Symbolic Manipulator Microworld is reported with an example of a symbolic transformation of the numerical expression of task 2. As we can see, the transformation of the expression is located in a working space on the left of the window while the list of commands to transform the expression is placed on its right. We note that these commands incorporate the properties of addition and multiplication and some rules for basic computation. These commands can be applied after the user has selected the part of expression they want to transform.
The selection through the mouse is under control of the system that provides two types of feedback:

- the first one is connected to the exploration of the hierarchical structure of the expression
- the second one is linked to the highlighting of commands that can be applied to the part of the expression selected by the user.

The feedback acts on the pragmatic, the cognitive, the social and the epistemic plane. As an example of mediation on the pragmatic plane let us consider the use of the command that transforms a number into a multiplication of its prime factors. Traditionally this operation needs time to be developed and it entails a shift of attention from the goal of the problem (demonstration of a property) to technical and repetitive computations. The use of the command allows students to focus on the conceptual aspects of task 2, delegating to the system the repetitive aspects of the computation.
As to the cognitive plane, we observe that the system offers different types of feedback able to support students in developing cognitive functions involved in the analysis and transformation of a numerical expression. For example, by moving the mouse on the expression it is possible to explore dynamically its hierarchical structure to support the recognition of the parts that can be transformed (Figure 6). As a matter of fact, when moving the mouse on symbols involved in the expression (brackets, numbers, letters, signs of operation) the system answers with a visual feedback which highlights the part of the expression affected by the symbol within the hierarchical structure of the expression.

$2\cdot\boxed{2}\cdot3\cdot5\cdot7+2\cdot2\cdot2\cdot3\cdot7+63$ The mouse points to the factor "2" of 2*2*3*5*7	$\boxed{2\cdot2\cdot3\cdot5\cdot7}+2\cdot2\cdot2\cdot3\cdot7+63$ The mouse points to an operator "* " of 2*2*3*5*7
$\boxed{2\cdot2\cdot3\cdot5\cdot7+2\cdot2\cdot2\cdot3\cdot7+63}$ The mouse points to an operator "+" of the whole expression	$2\cdot2\cdot\boxed{3\cdot5}\cdot7+2\cdot2\cdot2\cdot3\cdot7+63$ The mouse is dragged on the two elements of the selection

Figure 6: Dynamic exploration of the hierarchical structure of an expression.

After the selection of the part of expression to be transformed, the system answers with another feedback, i.e. it highlights and activates the applicable interface commands. This feature plays an important mediating role when students do not master properties to transform an expression and the effects that their application produces and also when they do not have a clear objective to reach with the transformation of the expression.

We note that the rules of the transformation and the effects they produce are distributed in the commands and in the interface of the Microworld and the student can exploit the cognitive support offered by the Microworld to solve the problem at hand. Students use a trial and error approach to become confident with the transformation of expressions. This explorative activity is crucial to favor the emergence of objectives for the task solution. The emergence of objectives for the solution of the task is the result both of the familiarization process with the rules of symbolic transformation and of processes of social negotiation able to highlight that the solution is performed when the form of the expression reflects the property indicated by the text of the task.

Finally, we observe that the Symbolic Manipulator Microworld mediates the transformation of the expression on the epistemic plane too. As a matter of fact, we note that in the Microworld this activity is more than a mere application of specific rules of computations (as more often than not it occurs in traditional practice). It can be considered as a demonstration activity of the equivalence of two symbolic expressions on the basis of a set of axioms that are reified and incorporated in the commands of the interface.

The commands of the interface, the effects produced by their application on a selected part of the expression and the whole transformation performed can be metaphorically used in the communication processes of the class to speak of axioms, mathematical theory, formal demonstration of a property and so on. In other words, they are useful concrete references to access to mathematical meanings that go beyond those involved in the task solution.

Conclusion

The two situations presented in this paper show that digital technology can be effectively exploited to construct new Microworlds able to support the development of capabilities in the solution of both concrete arithmetic problems and theoretical arithmetic problems. The Euro Microworld of ARI-LAB 2 makes available a digital numeric system characterized by new operative and representative potentialities that can be exploited to bring numerical procedures and concepts involved in the construction of the resolution for concrete arithmetic problems under the control of perception.

The Symbolic Manipulator Microworld makes available a digital symbolic manipulator characterized by new operative and representative potentialities able to bring under the control of perception the hierarchical structure of symbolic expressions, the rules that can be applied to an expression preserving the equivalence and the effects produced by their application.

We have highlighted that the mediation provided by these Microworlds to the learning process occurs on different planes (pragmatic, cognitive, epistemic and social) and this determines their educational relevance.

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