From String Art to Caustic Curves: Envelopes in Symbolic Geometry

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Abstract: *In this paper, we use the symbolic geometry program Geometry Expressions to analyze three problems involving envelope curves. First we examine the envelopes of families of lines passing through points which are equally spaced on a pair of line segments. We use a combination of symbolic geometry and algebra to develop an expression for the area of the void in a popular string art figure consisting of 3 parabolas inscribed in a triangle. We use an envelope approach to reduce a popular calculus problem - that of finding the longest ladder which fits around an asymmetric corner – to an algebra problem which is readily solved using CAS. Finally we study the caustic curves generated by reflecting a point light source in a shiny cylinder. We analyze these both experimentally and theoretically, and focus on determining the parametric and Cartesian locations of the cusps. These examples illustrate how symbolic geometry technology can be used to make mathematics fun, accessible and challenging.*

1. Introduction

The envelope is that phantom curve your eye picks out when you look at a family of straight lines. Perhaps the most familiar example of the envelope is found in string art, or “mathematical embroidery” [1], where a family of lines is created by attaching strings to artfully located sets of pins (Fig 1.). Envelopes also occur naturally as light caustics, here the family of lines is a family of reflected rays, and the caustic appears as a fringe of light.

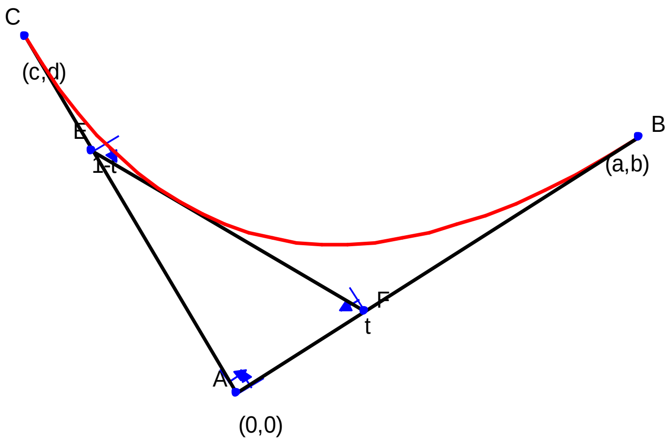
Geometry Expressions [4] (www.geometryexpressions.com) is a symbolic geometry system. As such it is able to generate symbolic expressions from geometric figures. Amongst other things, Geometry Expressions is able to generate the envelope curve of a family of lines or circles, and to derive both parametric and implicit equations for the curve. In this paper, we will use envelopes in the symbolic geometry program Geometry Expressions to explore a variety of mathematical problems ranging from string art to light caustics. We will note how the use of a symbolic geometry system in concert with a CAS allows a mode of thinking which is part geometric, part algebraic, and which facilitates problem solving, exploration and discovery.

2. String Art

Mathematically, the envelope of a one parameter family of lines is the curve which is mutually tangent to that family [3]. In Geometry Expressions a locus curve is specified by selecting a point and specifying the parameter which is to vary to create the locus. An envelope curve is created in a similar fashion except that instead of a point, a line, line segment or circle is selected.

Figure 1: We ask the question, what is the area of the void in the center of this string figure?

In Figure 2, we model a typical string art configuration where pins are evenly spaced along a pair of lines.
E is specified to lie proportion t along CA while F is proportion t along AB. The envelope of EF is constructed with respect to parameter t.



$$\Rightarrow b^2\cdot c^2-2\cdot a\cdot b\cdot c\cdot d+a^2\cdot d^2+Y^2\cdot\left[a^2+2\cdot a\cdot c+c^2\right]+X\cdot Y\cdot(-2\cdot a\cdot b-2\cdot b\cdot c-2\cdot a\cdot d-2\cdot c\cdot d)+X^2\cdot\left[b^2+2\right.$$

Figure 2: Geometry Expressions model of the envelope curve created by a string art model

The equation of the curve is second order and hence clearly a conic. Inspection of the coefficients shows that the second order term in Y and the term in XY disappear if a= -c. Hence, if points B and C are positioned equal distances on either side of the y axis, the parabola will be vertical (fig. 3)
Examining the parametric equation for the envelope (fig 3b), we see that X=0 occurs when the parameter t = ½. At this location the Y coordinate is $\frac{b+d}{4}$, or the midpoint of the median AG.

In the situation of figure 2, the triangle median AG is clearly vertical, as is the axis of symmetry of the envelope parabola. Hence, if we consider the general triangle (figure 1), we observe that the axis of symmetry of the parabola is parallel to the median of the triangle.

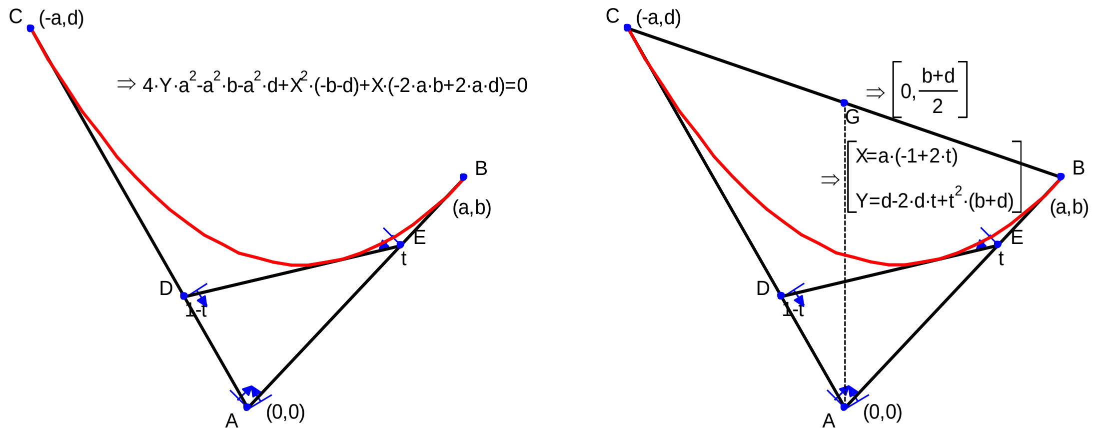


Figure 3: (a) Implicit equation of the envelope, where points B and C are equal distances along the x axis from A. (b) parametric equation of the envelope, along with the median AG of triangle ABC.

2.1 Areas

We address the question: what is the area of the region bounded by the three envelope curves in the string art of figure 1? We first ask the question: what is the area bounded by a single curve? We could do this using integration, but here we develop a geometrical argument. (This is essentially Archimedes method of parabola quadrature outlined in [2]).

It is convenient initially to transform our piece of string art so that the parabola is of the form $y=ax^2$ (fig 4).

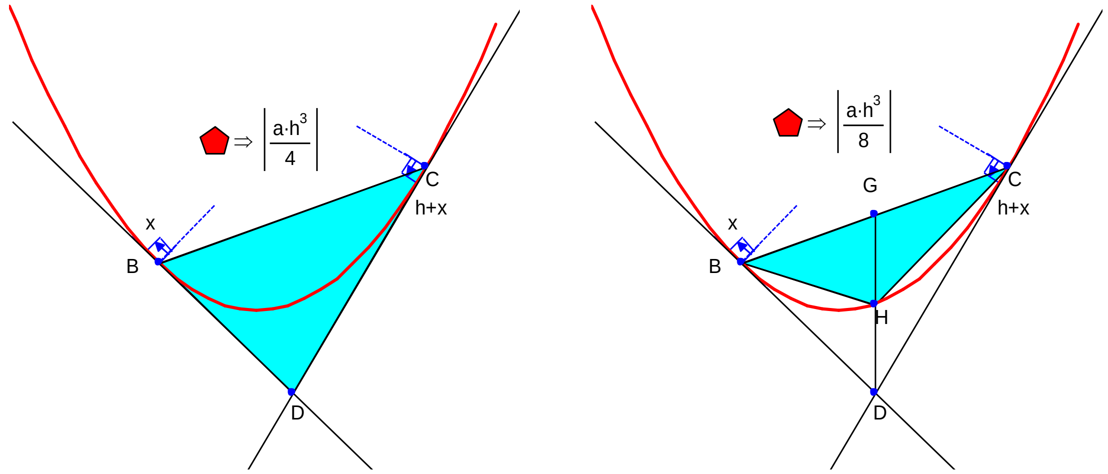


Figure 4: (a) Area of the triangle formed by a chord of the parabola and the tangents at its end depends only on the width of the chord, but not its location. (b) Area of BHC is half the area of BDC

Examining the triangle formed by the point on the parabola with x-coordinate x, the point with x-coordinate x+h, and the point at the intersection of the tangents to the parabola at those points, we see that its area is cubic in h, but independent of x.
If we now examine the triangle BHC, where H is the center of the median DG, we observe that its area is half that of BDC.

Now if we add a point I on the parabola at x coordinate $x+\frac{h}{4}$, the area BHI will be $\frac{ah^3}{64}$ and as the formula is independent of x, adding a point J at x location $x+\frac{3h}{4}$, the area HJC will also be $\frac{ah^3}{64}$.

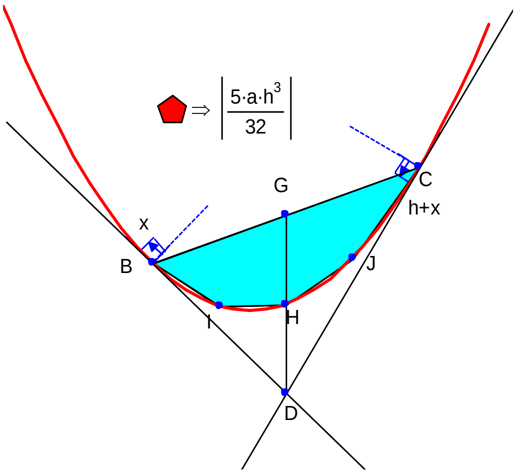


Figure 5: Area of BIHJC is 5/4 that of BHC

If the area of BHC is A, we see that the area of BIHJC is $A\left(1+\frac{1}{4}\right)$.

Continuing the process, we see that the area between the chord and the parabola is:

$$A\left(1+\frac{1}{4}+\frac{1}{16}+...\right)=\frac{4}{3}A$$

Now we look at three parabolas drawn on the sides of the same triangle. To calculate the points of intersection between the parabolas, we copy their equations into an algebra system and solve
We find 3 roots:

$$\{Y=0, X=0\}, \{Y=\frac{4\,c}{9}, X=\frac{4\,a}{9}+\frac{4\,b}{9}\}, \{$$

$$Y=\frac{4}{3}\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4)\,c, X=\frac{4}{3}($$

$$-a\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4)+5\,b\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4)$$

$$+5\,a-4\,b)\Big/(1+4\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4))\}$$

The second is the interesting one. From its form, we see that it is 8/9 of the way along the median. We can therefore draw the straight edged triangle covering this area by placing points 8/9 of the way along the medians (fig 6):

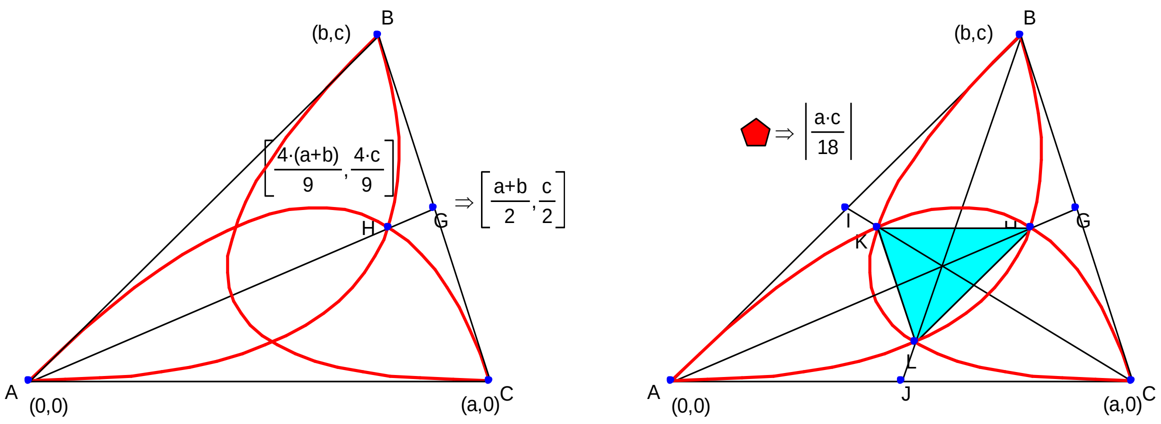


Figure 6: (a) The intersection of the parabolas is 8/9 of the way along the medians of the triangle. (b) Area KLM is 1/9 the area of ABC

As the area of the triangle ABC is ac/2, we see that the area of HKL is 1/9 the area of the original triangle. Now we can verify that K and L lie at equal distance to the median AG (fig 7).
Now let R be midpoint of the median AG. R is the point at which the parabola crosses the median. From the above, the area of the parabola segment KRL is 4/3 of the area of the triangle KLR, and hence is $\frac{ac}{81}$.

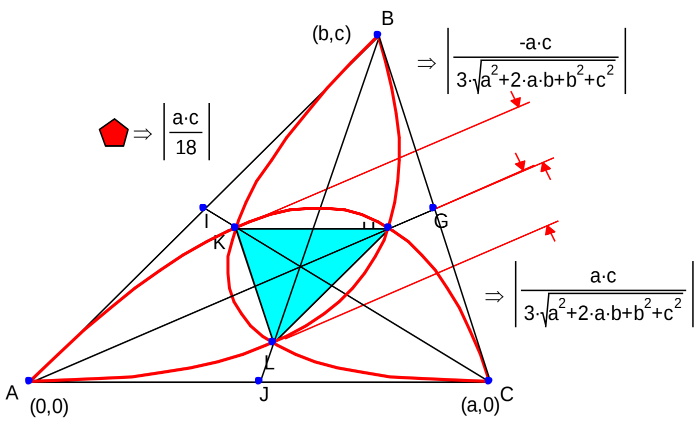


Figure 7: K and L lie at equal distances from the median AG.

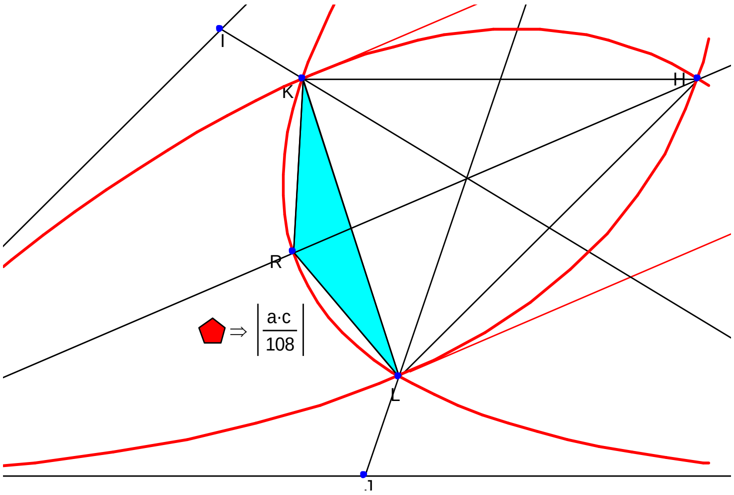


Figure 8: Segment KRL has area 1/54 that of the original triangle

Adding three such areas onto the area of the triangle HKL gives a total area for the void of:

$$V = \frac{5ac}{54}$$

or, if T is the area of the original triangle,

$$V = \frac{5T}{27}$$

3. A Ladder Problem

A classic problem, usually posed as an optimization problem in calculus, is that of finding the longest ladder which fits round a wall. Use of an envelope curve, however, can reduce this to a problem in algebra, readily solved with the aid of a CAS. It can be used to illustrate the relationship between geometric incidence between a point and a curve and the solution of an algebraic equation.

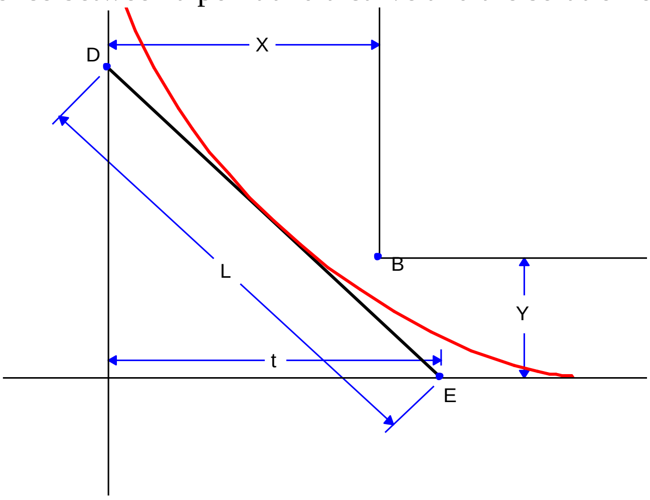


Figure 9: The envelope of a ladder of length L and the corner between corridors of width X and Y.

The problem is to determine the length of the longest ladder which will fit round a corner between two corridors of width X and Y. The situation is modeled in figure 9, and the envelope of the family of locations of the ladder is shown. To create this model in Geometry Expressions, a line DE

is drawn such that D is constrained to lie on the y-axis and E on the x-axis. The length of the line is constrained to be L, and the location of E is constrained to be distance t from the y-axis. The envelope curve is created simply by selecting the line and specifying which variable (t) should vary.
A key observation is that the ladder itself stays below its envelope curve, and therefore as long as the envelope curve does not intersect the corner, then the ladder will clear. In order to find the longest ladder which clears the corner, one wants to find the value for L such that the point B lies on the envelope curve.

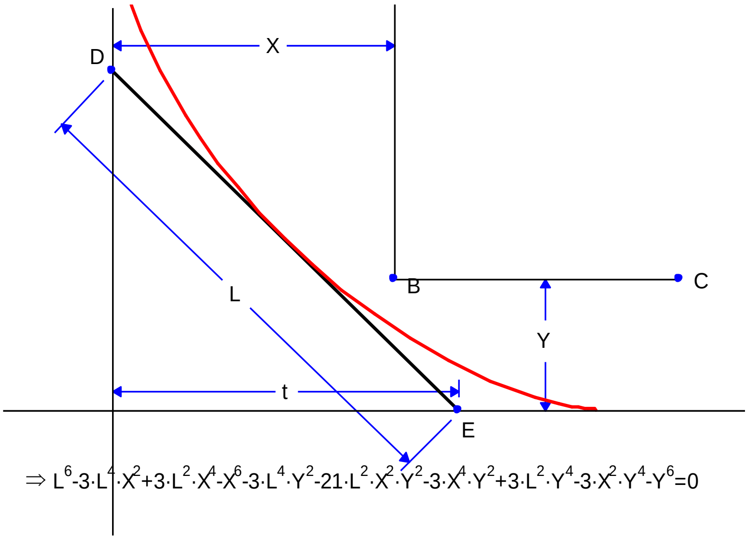


Figure 10: Equation of the envelope curve is computed by Geometry Expressions

Using Geometry Expressions, we can generate the equation of the envelope curve simply by selecting the curve and requesting the implicit equation. We notice that the curve equation is 6th order in L, but that it only includes even powers. Hence it can be considered as a cubic in L^2. We would therefore expect 6 solutions, but that they would be present as positive / negative pairs. Copying the equation from Geometry Expressions into an algebra system allows us to solve for L.

```
> solve(L^6-3*X^2*L^4+3*X^4*L^2-X^6-3*Y^2*L^4-21*Y^2*X^2*L^2-3*Y^2*X^4+3*Y^4*L^2-3*Y^4*X^2-Y^6 = 0,L);
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➢

$$\frac{\sqrt{(Y X^2)^{(1/3)} \left(Y^2 (Y X^2)^{(1/3)} + 3\, Y (Y X^2)^{(2/3)} + 3\, Y X^2 + X^2 (Y X^2)^{(1/3)}\right)}}{(Y X^2)^{(1/3)}}, \ldots$$

Four of the resulting solutions are complex, and of the remaining two real solutions, only one is positive. This is the one to copy back into Geometry Expressions. We observe that with this length ladder, the point B lies on the envelope curve and the ladder only just makes it around the corner.
As a final observation, we note that the solution for L is not symmetric in X and Y. However, the geometry problem clearly is symmetric: the longest ladder which can be dragged from a corridor of width X into a corridor of width Y is obviously the same length as the longest ladder which can be dragged from a corridor of width Y into one of width X. It would therefore seem reasonable that some simplification could be done on the solution to expose its essential symmetry. This could be set as an exercise to be completed by hand, or one could coerce simplification in the algebra system by asserting that both X and Y are positive.

```
> simplify(%) assuming X>0, Y>0;
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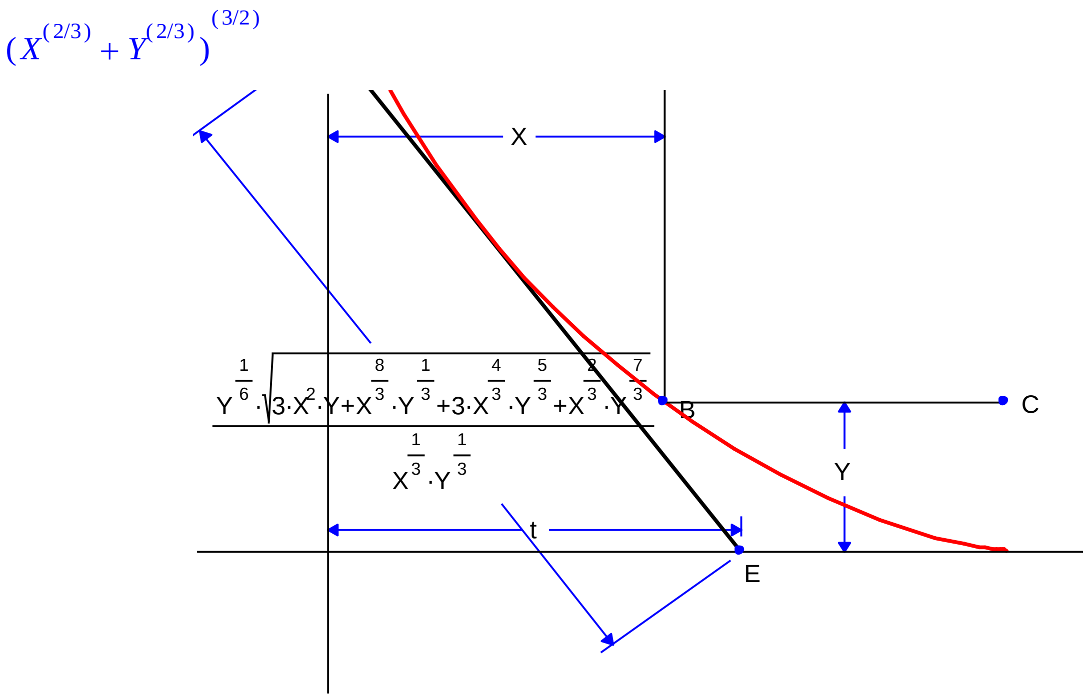

Figure 11: Solution is pasted back from Maple into Geometry Expressions, and we observe that the corner point B lies on the envelope.

4. Light Caustics

When light reflects off a convex curved surface, the reflected rays form a curve of bright light called a caustic. This curve is exactly the envelope curve of the reflected rays. Figure 12 shows the caustic curve formed by light reflecting in a wedding ring.

Figure 12: Caustic curves formed by light reflecting inside a gold ring

We will study the form of the caustic curve, when light from a point source reflects in a cylinder. To acquire experimental data against which to compare our theoretical findings, we use the apparatus shown in figure 13. A Perspex cylinder is used as our reflecting surface. A small LED lamp is used to position our light source inside the cylinder. A circular paper disk is marked in concentric rings, each ring being 1/10 of the radius of the cylinder. This allows us to observe the location of features of the curve as a proportion of the cylinder radius.

Figure 13: Apparatus for studying caustic curves with a light source inside the reflecting cylinder.

In figure 14 we see the caustic curve when the light source is at the circumference of the cylinder, and when the light source moves closer into the center. A video of this experiment may be seen at: http://www.youtube.com/watch?v=UBSu7NJiI3c

Figure 14: Caustic curves observed with (a) the light at the circumference of the cylinder and (b) the light source interior to the cylinder.

We can model the behavior of a single beam of light in Geometry Expressions. First we create a circle AB, and constrain its center to be (0,0) and its radius to be 1. We then constrain the parametric location of B on the curve to be t. This has the effect of specifying the line AB to be angle t (in radians) from the x axis. Geometry Expressions allows you to reflect in a line, but not in a curve. However reflection of light in a cylinder is equivalent to reflection in the tangent to the cylinder. Hence we draw a line through B and constrain it to be tangent to the circle. We now create point C, constrain its location to be (0,-a), and draw an infinite line through C and B. We complete the model by reflecting BC in the tangent line.

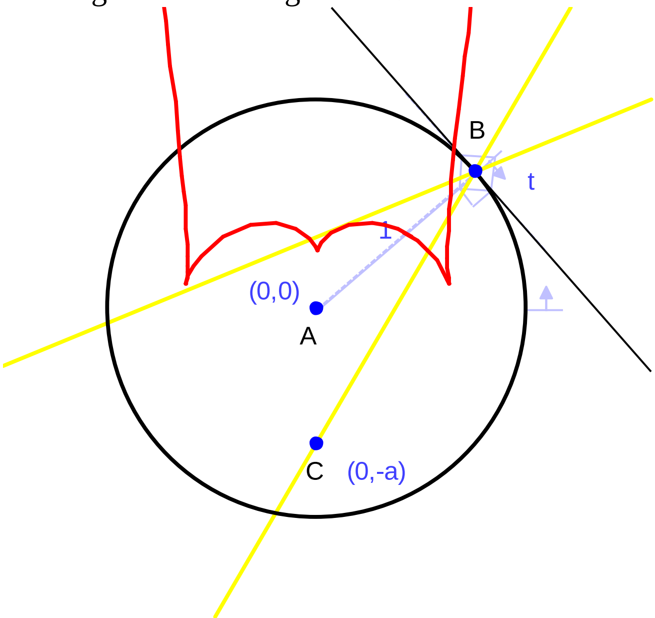

Figure 15: Geometry Expressions model of a beam of light reflected in a cylinder.

The caustic curve is the envelope of the family of reflected lines. We are interested in the location of the central cusp. First in the situation where a = 1 (fig. 16)

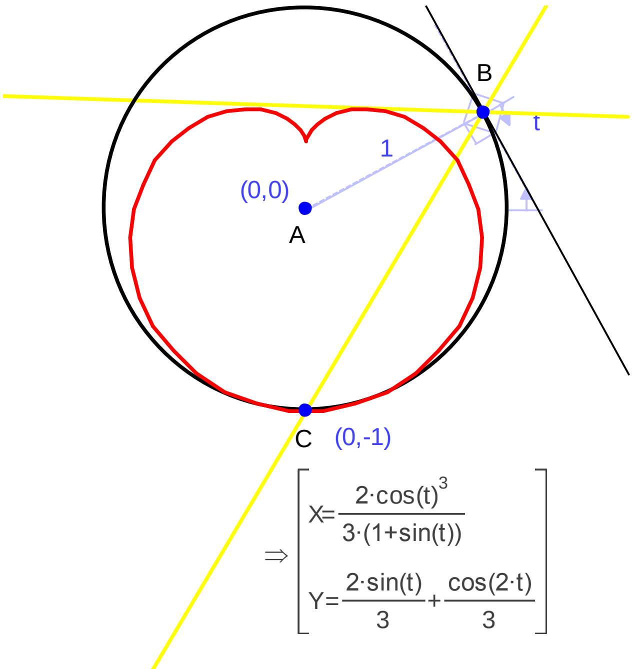

Figure 16: Parametric equation of the caustic with light source on the circumference

The parameter t specifies the location of point B on the circle as an angle in radians. When B is at the location (1,0) t will have the value 0, when B is at the location (0,1) t will have the value $\pi/2$, etc. Clearly when B is vertically above A, the light from C will reflect straight back, and will touch the curve at the cusp.
To find the location of the cusp, we need simply substitute t= $\pi/2$ into the Y component of the curve equation in figure 16:

$$\frac{2\sin\frac{\pi}{2}}{3}+\frac{\cos\pi}{3}=\frac{1}{3}$$

Looking at the picture in figure 14a, we see that the light cusp is somewhere around the third circle or approximately 0.33 radii from the center

4.1 Cusp on the Circumference

As the light source moves towards the center of the cylinder, the caustic curve is no longer fully contained in the cylinder. However, at some point, it re-enters to give the shape in Figure 17. Using the circular graph paper, we can estimate the location of the central cusp at this point to be on the second circle.

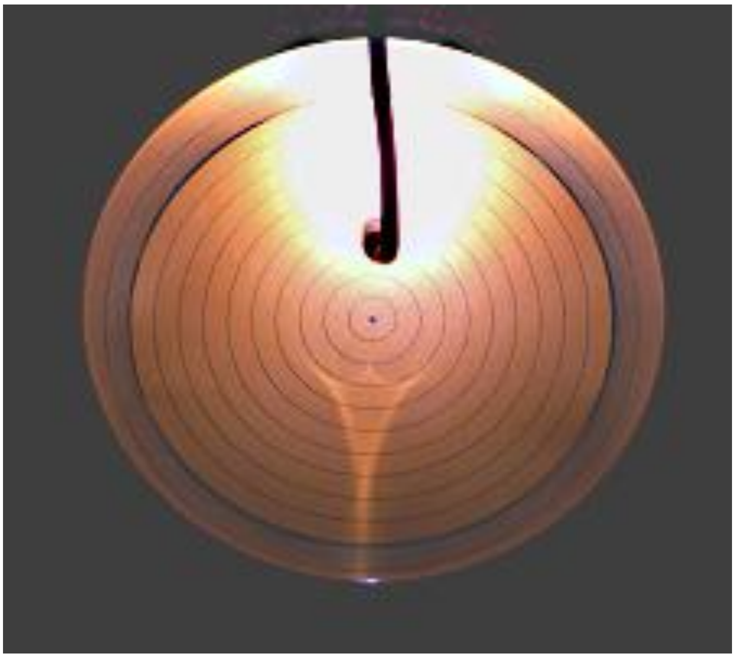

Figure 17: The caustic when the far cusp touches the circumference.

In Geometry Expressions, we can change the coordinates of C back to (0,-a) and drag C towards the center of the circle until the second central cusp appears:

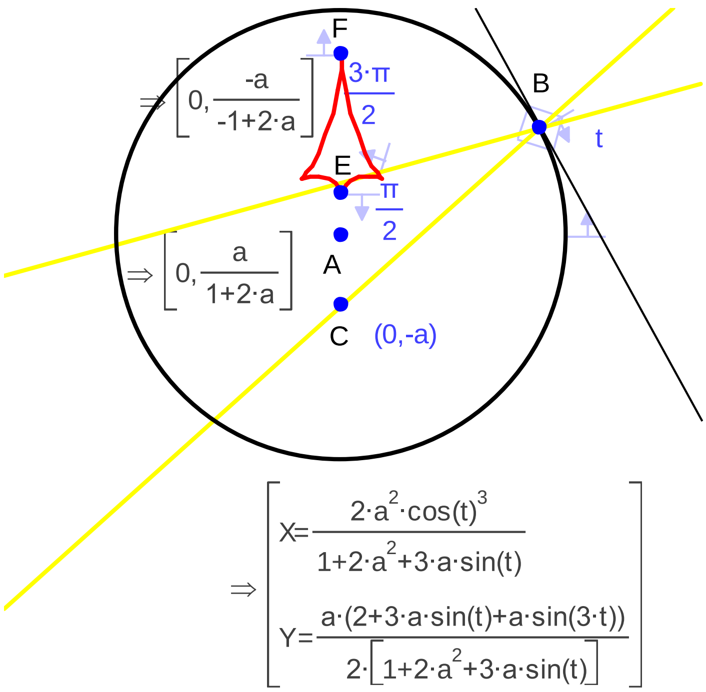

Figure 18: Curve equations for general a. Points E and F are put on the curve and constrained to be at parametric locations π/2 and 3π/2

In figure 18, points E and F are positioned at parametric locations π/2 and 3π/2 on the curve. Coordinates for these points have been computed by the software.
For F to lie on the circumference, we need:

$$\frac{-a}{-1+2a}=1$$

Solving for a yields:

$$a=\frac{1}{3}$$

Substituting into the expression for the y coordinate of E gives

$$\frac{\frac{1}{3}}{1+\frac{2}{3}}=\frac{1}{5}$$

Examination of figure 17 shows the cusp lies on the second ring of the circular graph paper, or 1/5 of the way out from the center to the circumference, which corresponds exactly with our theoretical predictions.

4.2 Further Questions

Let's assume we move our light source a long way from the cylinder, so that the distance a is effectively infinite. Our cusp will now have location:

$$\lim_{a\to\infty}\frac{a}{1+2a}=\frac{1}{2}$$

This is the situation observed in the wedding ring (fig. 12).
Having analyzed the central cusps, we are left with the question what are the coordinates of the cusps which do not lie on the y axis?
For a curve (X(t),Y(t)), cusps occur where the derivatives of X and Y with respect to t simultaneously vanish [3]. The caustic's curve equation can be copied into Maple, differentiated and solved as follows:

$$X := \frac{2\cos(t)^3 a^2}{1 + 2a^2 + 3\sin(t)\,a}$$

$$Y := \frac{(2 + 3\sin(t)\,a + \sin(3\,t)\,a)\,a}{2\,(1 + 2a^2 + 3\sin(t)\,a)}$$

```
> solve({diff(X,t)=0,diff(Y,t)=0},t);
```

$$\left\{t = -\frac{1}{2}\pi\right\}, \left\{t = \frac{1}{2}\pi\right\}, \{t = \arctan(-a, \mathrm{RootOf}(a^2 - 1 + \_Z^2))\}$$

```
> allvalues(arctan(-a, RootOf(a^2-1+_Z^2)));
```

$$\arctan(-a, \sqrt{1-a^2}), \arctan(-a, -\sqrt{1-a^2})$$

Drawing points on the curve with these parametric values (copying the solutions back from Maple into Geometry Expressions), we see they lie on the cusps (fig. 19a).

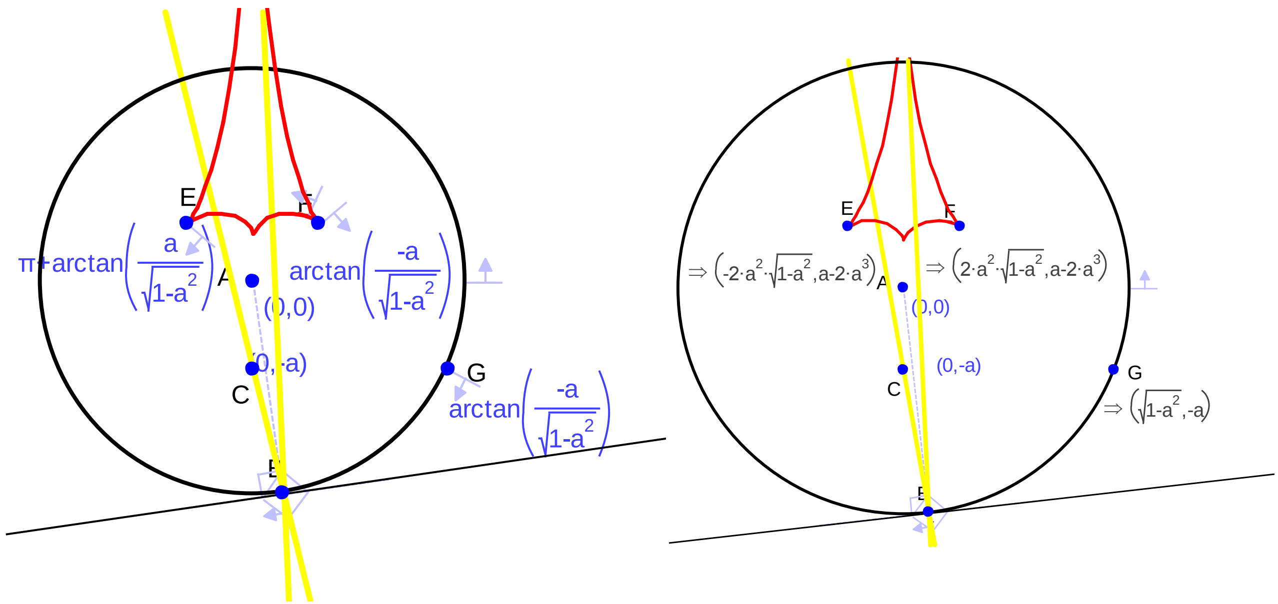

Figure 19: (a) G is the point on the circumference of the circle with the same parametric location as the right hand cusp. (b) G has the same y coordinate as C. Coordinates of the cusps are also given.

Where does point G lie on the circle when the reflected ray goes through the cusp? One way of finding out is to put a point on the circle and give it the arctan() as its value. Now, hiding the parametric locations of E,F and G, we can display their coordinate locations (fig 19b). We see that G has the same y coordinate as C.

We have discovered that cusps occur when the original ray emanates from point C either in a vertical or a horizontal direction. The vertical seems geometrically intuitive, this is the direction at which the ray reflects back on itself, and one might expect singular behavior. What, if anything, is special about the situation where the original ray is horizontal?

To answer this question, we consider the angle between the incident ray and the reflected ray. Clearly when the incident ray is vertical, this angle is 0 and hence at a minimum. We ask the question, what direction for CB yields a maximum angle?

First, we note that this is equivalent to finding a maximum value for CBA. We could, of course generate an equation for this angle in Geometry Expressions, then differentiate and solve to yield the appropriate value for t:

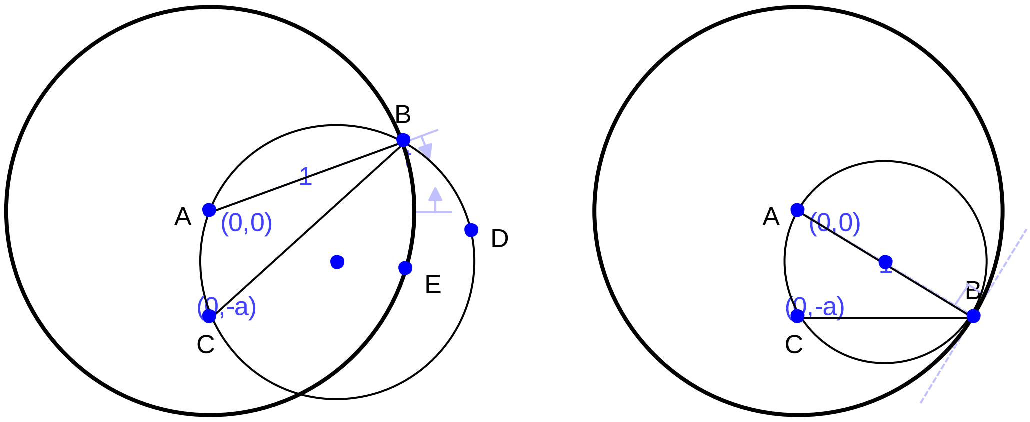

Figure 20: (a) E is inside the circumcircle of ABC, hence angle AEC < angle ABC. (b) The circumcircle of ABC is tangent to the original circle. This corresponds to a maximum of angle ABC.

However, a geometric approach to the problem can be taken by observing that all the points D such that ADC = ABC lie on the circumcircle of ABC. In figure 20a we observe that if this circumcircle intersects the original circle in two places, and if E lies on the portion of the circumference of AB cut off by it then AEC>ABC. Hence, ABC can only be maximal if the circumcircle to ABC is tangential to circle AB (fig 20b). In this case it is easy to see that AB is a diameter of the circumcircle and hence the angle ACB is right.

We have hence shown that the cusps of the caustic occur at extrema of the angle between incident and reflected rays.

5. Conclusion

We have examined three situations where the automatic derivation of envelope curves within a symbolic geometry program (Geometry Expressions) allow problems which would normally be addressed using calculus to be analyzed using a combination of geometry and algebra. In addition,

results obtained symbolically, using a computer algebra system, motivated further geometrical study.
Without the use of technology, envelope curves are not accessible at a high school level. With the use of technology they can become a rich source of example problems and illustrative scenarios. In short, we have made some interesting and fun mathematics accessible through technology.

Supplementary Documents

Geometry Expression file for experimentation in Section 2
Geometry Expression file for experimentation in Section 3
Geometry Expression file for experimentation in Section 4

References

[1] D. Pedoe (1976) Geometry and the Visual Arts, Dover, New York
[2] H. Dorrie (1965) 100 Great Problems of Elementary Mathematics, Dover, New York
[3] DJ Struik (1950) Lectures on Classical Differential Geometry, Dover, NY
[4] Geometry Expressions website www.geometryexpressions.com